A COMPLETE DETERMINATION OF THE COMPLEX QUADRATIC FIELDS OF CLASS-NUMBER ONE

H. M. Stark

1. INTRODUCTION

Let $h(d)$ be the class-number of the quadratic extension of the rationals of discriminant d. In the case of complex quadratic fields, it has long been known that

$$h(-p) = 1 \qquad \text{for } p = 3,\ 4,\ 7,\ 8,\ 11,\ 19,\ 43,\ 67,\ 163\,.$$

Heilbronn and Linfoot [5] have shown that there is at most one more value of p for which $h(-p) = 1$, and the author has shown [7] that if such an additional value p exists, then $p > \exp(2.2 \cdot 10^7)$. Heegner [3] attempted to show that the above nine values of p give the only complex quadratic fields of class-number 1. Unfortunately, it is thought that there is a gap in Heegner's proof, possibly traceable to one of his references, Weber [8]. Nevertheless, it will be instructive to examine briefly Heegner's method.

Put

$$f(\omega) = q^{-1/24} \prod_{\nu=1}^{\infty} (1 + q^{2\nu-1}),$$

where $q = e^{\pi i \omega}$ and $\Im\omega > 0$. Also, put $\gamma_2(\omega) = f(\omega)^{16} - 16\, f(\omega)^{-8}$. It is known [8, Section 125] that when $p \equiv 3 \pmod 4$ and $h(-p) = 1$, $\gamma_2\left(\frac{-3+\sqrt{-p}}{2}\right)$ is a rational integer. Thus for $p \equiv 3 \pmod 4$ and $h(-p) = 1$, $f\left(\frac{-3+\sqrt{-p}}{2}\right)$ satisfies an equation of degree 24 with integral coefficients,

$$x^{24} - \gamma_2 x^8 - 16 = 0\,.$$

Heegner reduces this equation to one of degree 12,

$$x^{12} + 2\zeta x^8 + 2\zeta^2 x^4 - 4 = 0\,,$$

with the relation

$$-4\zeta(\zeta^3 + 4) = \gamma_2\,.$$

This equation in turn is reduced to an equation of degree 6,

$$x^6 + 2\alpha x^4 + 2\beta x^2 - 2 = 0\,,$$

with the relations

$$\zeta = 2(\beta - \alpha^2), \qquad \zeta^2 = 2(\beta^2 + 2\alpha)\,.$$

These relations can be combined to give the equation

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