# MONOTONE FUNCTIONS AND CONVEX FUNCTIONS

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In their note [1], A. S. Besicovitch and R. O. Davies prove the following interesting theorem concerning monotone functions of a real variable.

THEOREM. *Let* $f$ *be a real-valued, nonnegative, continuous, monotone function defined on* $I = [0, 1]$. *Then there exist two convex functions* $g_1$ *and* $g_2$ *on* $I$ *such that* $0 \leq g_1 \leq f \leq g_2$ *and*

$$2 \int_0^1 g_1 \, dx \geq \int_0^1 f \, dx \geq \frac{1}{2} \int_0^1 g_2 \, dx .$$

*Furthermore, the constants* 2 *and* 1/2 *are best possible.*

In the present note, we extend this theorem to functions that are monotone on $I^n$, the n-fold cartesian product of I. Of course, the constants involved will depend upon n. The extension is an immediate corollary of Theorem A below, which concerns the measure of a certain family of subsets of euclidean n-space. The present proof is not an extension of that in [1], and in a sense it is better, since it avoids a transfinite construction.

## 1. THE STATEMENT OF THE MAIN THEOREM.

By R we mean the real numbers, and by P the nonnegative real numbers. $X^n = X \times \cdots \times X$ will denote the n-fold cartesian product of a set X.

For each point $x = (x^1, x^2, \cdots, x^n) \in P^n$, let

$$B_n(x) = \{y \in R^n \mid 0 \leq y^i < x^i \ (i = 1, 2, \cdots, n)\},$$

$$\mathscr{M}_n = \{M \subset R^n \mid M \text{ is a bounded open set such that } x \in M \cap P^n \text{ implies } B_n(x) \subset M\}.$$

We shall partially order $R^n$ by agreeing that $x \geq 0$ if and only if $x^i \geq 0$ $(i = 1, 2, \cdots, n)$. For each point $x \in R^n$ such that $x^i > 0$ $(i = 1, 2, \cdots, n)$, we let

$$K_n(x) = \{y \in R^n \mid \sum_{i=1}^{n} (y^i/x^i) \geq n\}.$$

Corresponding to each $M \in \mathscr{M}_n$, we define three sets $O_n(M)$, $C_n(M)$, and $H_n(M)$ as follows:

Received June 8, 1965.

The first author was supported by National Science Foundation Research Grant GP 3834.