ON THE EXISTENCE OF ALMOST PERIODIC MOTIONS

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In the first part of this paper we investigate the question of necessary and sufficient conditions for the existence of (Bohr) almost periodic motions in dynamical systems. The results are applied, in the second part, to give an existence criterion for almost periodic solutions of ordinary differential equations.

PART I. DYNAMICAL SYSTEMS

V. V. Nemyckiĭ and V. V. Stepanov [7] investigated conditions under which the ω-limit set of a positively Lagrange-stable motion is a minimal set of almost periodic motions. They proved that a sufficient condition for this to hold is that 1) the motion approximates the ω-limit set uniformly and 2) the motion is uniformly positively Lyapunov-stable with respect to the positive semitrajectory. They remarked that the question of necessary conditions is still open.

In this part of the paper we present another sufficient condition (Theorem 5) that the ω-limit set of a positively Lagrange-stable motion be a minimal set of almost periodic motions. It will be seen that our condition (which is stronger than that of Nemyckiĭ and Stepanov) is more readily applicable. The question of a necessary condition remains open. However, we are able to give a partial solution to this problem.

Let (X, d) be a metric space with a dynamical system π: $X \times R \to X$, where R denotes the real numbers. Let Ω_p denote the ω-limit set of the point $p \in X$. Let $\gamma^+(p) = \{\pi(p, t): t \geq 0\}$ denote the positive semitrajectory and $\gamma(p) = \{\pi(p, t): t \in R\}$ the trajectory of the motion through p. If $I \subset R$, then $\pi(p, I)$ will denote the set $\{\pi(p, t): t \in I\}$. If $A \subset X$ and $\varepsilon > 0$, then the ball about A of radius ε is given by

$$\mathfrak{B}(A; \varepsilon) = \{p \in X: d(p, A) < \varepsilon\}.$$

A motion $\pi(p, t)$ is said to be *recurrent* if for every $\varepsilon > 0$ there is a $T > 0$ such that for each $t_0 \in R$

$$\gamma(p) \subset \mathfrak{B}(\pi(p, [t_0, t_0 + T]); \varepsilon).$$

The motion $\pi(p, t)$ is *Lagrange-stable* if $Cl\gamma(p)$ is compact, and it is *positively Lagrange-stable* if $Cl\gamma^+(p)$ is compact. It is known [7, Theorem 7.09] that a Lagrange-stable motion $\pi(p, t)$ is recurrent if and only if for every $\varepsilon > 0$ the set $\{\tau \in R: d(p, \pi(p, \tau)) < \varepsilon\}$ is relatively dense in R. A motion $\pi(p, t)$ is said to be (Bohr) *almost periodic* if for every $\varepsilon > 0$ the set

$$\{\tau \in R: d(\pi(p, t), \pi(p, t + \tau)) < \varepsilon \text{ for all } t \text{ in } R\}$$

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