CHARACTERISTIC NUMBERS AND HOMOTOPY TYPE

P. J. Kahn

1. INTRODUCTION AND STATEMENT OF RESULTS

Let Ω denote the oriented cobordism ring (see [10]), and [M] the oriented cobordism class of the C^∞-manifold M, which we assume to be closed, compact, and oriented, but not necessarily connected. Ω is graded by manifold dimension. In Ω_n, let I_n denote the set of all classes of the form $[M] - [M']$, where M and M' are n-manifolds of the same oriented homotopy type. It is clear that I_n is a subgroup of Ω_n and that the graded group $I = (I_0, I_1, I_2, \cdots)$ is an ideal in Ω.

The following result follows easily from the definitions and from certain elementary facts about Pontrjagin and Stiefel-Whitney numbers.

THEOREM 1. *I_n is a free abelian group. If* $n \not\equiv 0 \pmod 4$, *then* $I_n = 0$. *If* $n \equiv 0 \pmod 4$, *then* co-rank $I_n \geq 1$, *where* co-rank $I_n = \operatorname{rank} (\Omega_n/I_n)$.

Note that since $\Omega_4 \simeq Z$ (this is well-known), Theorem 1 implies that $I_4 \simeq 0$.

Atiyah and Hirzebruch prove, in [1], that Pontrjagin classes are homotopy invariants (mod 8). We use this to prove the following assertion.

THEOREM 2. *The members of* I_n *are divisible by* 8.

The results of [5]—see the proof of Theorem 3 in Section 3, below—imply the following.

THEOREM 3. $\Omega \otimes Q \simeq Q[Y_4] \oplus (I \otimes Q)$.

(Explanation of notation: Q denotes the field of rational numbers, $Q[Y_4]$ denotes the polynomial ring over Q generated by some $Y_4 \in \Omega_4 \otimes Q$, and the symbol $\oplus$ denotes vector-space direct sum.)

COROLLARY 3.1. $I \otimes Q$ *is a prime ideal in* $\Omega \otimes Q$.

COROLLARY 3.2. co-rank $I_{4k} = 1$.

COROLLARY 3.2.1. *There is, up to a rational multiple, only one homotopy-invariant rational linear combination of Pontrjagin numbers (the* L_k*-genus* (see [4, p. 13]), *being such a combination).*

In [9], Tamura constructs certain 8-manifolds representing nontrivial elements of I_8; in [5], we extend his results to dimension 12. This enables us to obtain partial information about generators for I_8 and I_{12}.

THEOREM 4. *Let* X_i *denote the class in* Ω_{4i} *of complex projective* 2i*-space* $(i = 1, 2, 3)$, *and let* $A = X_2 - X_1^2$, $B = X_3 - X_2X_1$. *Then*

(i) I_8 *is generated by* $2^n \cdot 48A$, *for some integer* n $(0 \leq n \leq 3)$, *and*

(ii) I_{12} *has rank* 2 *and contains* $384X_1A$ *and* 576B; *all elements of* I_{12} *are of the form* $rX_1A + sB$, *where* $r \equiv 0 \pmod{24}$ *and* $s \equiv 0 \pmod{72}$.

Received September 8, 1964.

While preparing this paper, the author was supported by NSF Contract GP-2497.