ON ANALYTIC CONTINUATION OF LAURENT SERIES

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1. INTRODUCTION

There are various theorems, dating back to 1900, concerning analytic continuation of power series $\Sigma_0^\infty a_n z^n$ in which $a_n = g(n)$ for a holomorphic function g of some sort. Here we shall discuss certain extensions and analogs of the theorems of Wigert, Hardy, and Kronecker, which can be stated as follows.

THEOREM OF WIGERT [8, p. 288]. *$\Sigma_0^\infty a_n z^n$ defines a holomorphic function which has 1 as its only possible singularity and which vanishes at ∞ if and only if there is an entire function g of exponential type 0 such that* $g(n) = a_n$ $(n = 0, 1, 2, \cdots)$.

THEOREM OF HARDY [5, p. 338]. *Let $0 < \rho < \pi$, and let S be the circle $|z| = \rho$. In order that a series $\Sigma_0^\infty a_n z^n$ define a function holomorphic in the exterior of the curve e^{-S}, having a singularity on e^{-S}, and vanishing at ∞, it is necessary and sufficient that there exists an entire function g of exponential type ρ such that* $g(n) = a_n$ $(n = 0, 1, 2, \cdots)$.

THEOREM OF KRONECKER [4, p. 321]. *$\Sigma_0^\infty a_n z^n$ defines a rational function if and only if the infinite matrix (a_{i+j}) has finite rank.*

We begin by obtaining—Theorem 1 below—a complete generalization of the theorem of Hardy, in the sense that we are able to consider functions f that are holomorphic in the complement of an arbitrary compact set. The "coefficient functions" g that then occur are entire functions of arbitrary exponential type. However, the Laplace transform $\hat{g}$ of g can be continued holomorphically to a region whose complement B is bounded and has the property that $\bigcup_{-\infty}^{\infty}(B + 2n\pi i)$ does not separate the plane. The function f is then holomorphic in the complement of e^{-B}. It is clear from Pólya's theory of the indicator diagram (see [7] or [3, pp. 66-77]) that the above property of $\hat{g}$ is weaker than Hardy's type-restriction on g. Another feature of Theorem 1, important for the subsequent discussion of Laurent series, is that the expansions of f about 0 and ∞ are treated symmetrically.

Our results on Laurent series—Theorems 2 and 3—are concerned with a pair of series $\Sigma_{-\infty}^{\infty} a_n z^n$ and $\Sigma_{-\infty}^{\infty} b_n z^n$ that converge in two disjoint annuli. Theorem 2 gives a necessary and sufficient condition on the sequence $\{a_n - b_n\}$ in order that the given series be Laurent expansions of the same holomorphic function. Theorem 3 gives the condition that this be the case and that there be only poles and essential singularities between the two annuli. It is also mentioned here how the location of the singularities and the construction of the principal parts depend upon the sequence $\{a_n - b_n\}$. Finally, to illustrate Theorem 3, we derive a formula from the theory of elliptic functions.

Notation and terminology. $\mathfrak{C}$ will denote the complex plane. ∂E will denote the topological boundary of E $(E \subset \mathfrak{C})$. A region is a nonempty, open, connected subset of $\mathfrak{C}$. Holomorphic means analytic and single-valued in a region that may or may not be specified. If g is an entire function of exponential type, $\hat{g}$ denotes its Laplace

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