AN ARCHIMEDEAN PROPERTY OF CARDINAL ALGEBRAS

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A partially ordered group is called *Archimedean* if $na \leq b$ for all integers n implies $a = 0$, and *integrally closed* if $na \leq b$ for all $n \geq 1$ implies $a \leq 0$ [1, pp. 225, 229]. The latter property, expressed in terms of positive elements, reads as follows:

$$na \leq nb + c \text{ for all } n \geq 1 \quad \text{implies} \quad a \leq b.$$

Because of the presence of infinite elements, this property fails in every nontrivial cardinal algebra. It is the purpose of this note to show that every cardinal algebra satisfies the following closely related condition:

$$na \leq nb + c \text{ for all } n \geq 1 \quad \text{implies} \quad a + c \leq b + c.$$

The study of such properties was undertaken in the hope (as yet unfulfilled) of shedding some light on the simple cardinal algebras.

A *cardinal algebra*, as defined by Tarski [2], is an algebraic system consisting of a set A, a binary operation $+$ on A, and an operation Σ of countably infinite rank on A satisfying axioms which assert closure under the operations, the existence of a zero element 0, unrestricted commutativity and associativity of the operations, that $+$ is the restriction of Σ to two nonzero summands, and the validity of the following two principles:

Refinement. If $a + b = \Sigma c_i$, then there exist $a_i, b_i \in A$ such that $c_i = a_i + b_i$ for all $i < \infty$, $a = \Sigma a_i$, and $b = \Sigma b_i$.

Remainder. If $a_n = b_n + a_{n+1}$ for all $n < \infty$, then there exists $c \in A$ such that $a_n = c + \Sigma_i b_{n+i}$ for all $n < \infty$. (Summation indices are to run over the natural numbers, and the phrase "all $n < \infty$" refers to the set of natural numbers.)

A cardinal algebra can be partially ordered by defining $a \leq b$ to mean $a + x = b$ for some $x \in A$; this is the order referred to above. We begin by listing the results from [2] that we shall need, the numbering being that of [2]:

1.29. $a + b = b$ if and only if $\infty a \leq b$.

2.10. If $a + nc \leq b + (n+1)c$ for some $n < \infty$, then $a \leq b + c$.

2.21. If $a_0 + a_1 + \cdots + a_n \leq b$ for all $n < \infty$, then $\Sigma a_i \leq b$.

2.24. Every increasing sequence of elements of A has a least upper bound in A.

2.28. If $a_m \leq b_n$ for all $m, n < \infty$, then there exists $c \in A$ with $a_m \leq c \leq b_n$ for all $m, n < \infty$.

2.29. If $a \leq b_n + c$ for all $n < \infty$, then there exists $d \in A$ with $a \leq d + c$ and $d \leq b_n$ for all $n < \infty$.

2.33. If $na \leq nb$ for some $n < \infty$, then $a \leq b$.

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