CONDITIONS FOR THE ANALYTICITY OF CERTAIN SETS

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1. INTRODUCTION

In complex analysis it is not always easy to recognize an analytic function or an analytic set. The problem of recognizing an analytic function is a special case of the problem of recognizing an analytic set, since a function is analytic if and only if its graph is an analytic set.

In the first part of this paper it will be shown that a limit of a sequence $\{A_i\}$ of analytic sets of pure dimension k is an analytic set, provided that the 2k-dimensional volumes of the sets A_i are uniformly bounded. This generalizes a result of Stoll [9].

In the second part of this paper we discuss sets that are analytic except for possible singularities. The question is then whether the singularities are genuine or removable. Conditions guaranteeing the removability of singularities have been given by many authors, including Hartogs, Radó, Thullen, Remmert and Stein, Rothstein, and Stoll. Our results extend theorems of Stoll [8], [9]. We show that an analytic set A of pure dimension k defined in the complement of an analytic set B can always be continued through B, in case the 2k-dimensional volume of A is finite or $\bar{A} \cap B$ has 2k-dimensional Hausdorff outer measure 0. The first of these results was conjectured by Stoll, and it can be applied to give a simple proof of Stoll's theorem that an analytic subset of $\mathfrak{C}^n$ is algebraic if its volume of appropriate dimension doesn't grow too fast near infinity. Along the way we give simple proofs of the theorems of Radó and of Remmert and Stein, and derive some interesting properties for representing measures in certain algebras of analytic functions.

In the last section we introduce a general notion of capacity and use it to prove a very general extension of the theorem of Remmert and Stein.

2. CONVERGENCE OF ANALYTIC SETS

Rutishauser [7] gives a remarkable lower bound on the length of the curve in which an analytic subset of $\mathfrak{C}^2$ that passes through the origin intersects the unit sphere. The following result, communicated with its proof to the author by G. Stolzenberg, generalizes Rutishauser's. Rather than the precise lower bound $2\pi r$, we give here only the more easily proved lower bound r, since the precise result will be a consequence of Theorem 2.

LEMMA 1. *Let* B *be an open ball of radius* r *centered at* 0 *in* $\mathfrak{C}^n$, *and* A *a one-dimensional analytic set in some neighborhood of* $\bar{B}$, *with* $0 \in A$. *Let* S *be the boundary of* B. *Then the curve* $A \cap S$ *has length at least* r.

Proof. Let μ be any measure on $A \cap S$ that represents 0. By this we mean that μ is a nonnegative Baire measure on $A \cap S$ for which

$$(*) \qquad f(0) = \int f \, d\mu$$

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