ON THE CENTROID OF A HOMOGENEOUS WIRE

G. D. Chakerian and S. K. Stein

1. INTRODUCTION

Let C be a closed convex curve in the Euclidean plane E_2. If C has continuous curvature, then the *curvature centroid* of C is defined as the center of mass of C considered as a wire whose density is equal to the curvature at each point. Hayashi [5] shows that at least four normals of C pass through its curvature centroid. Bose and Roy [2] and Tietze [6] prove that the *area centroid* of C has the same property (the area centroid is the center of mass of a disk of uniform density bounded by C). In this paper, we prove that the *perimeter centroid* of C also has this property (Section 4). (The point (x_0, y_0) is the perimeter centroid of C if

(1) $$Lx_0 = \int_C x\,ds, \qquad Ly_0 = \int_C y\,ds,$$

where L is the length of C, and s is arc length along C.) The proofs in [2] and [5] employ Fourier series and put restrictions on the smoothness of C; however, even if C is assumed smooth, this technique fails to give the result for the perimeter centroid. Indeed, Bose and Roy [3] obtain by these methods only the weaker result that if m is the number of points on C where the radius of curvature is equal to three times the support function with respect to (x_0, y_0), and n is the number of normals through (x_0, y_0), then $m + n \geq 4$. The proof we give in Section 4, like that of Tietze [6] in the case of the area centroid, is purely geometric and places no smoothness restrictions on C.

The authors are indebted to the referee for several helpful suggestions.

2. DEFINITIONS

A *support line* of C is a line intersecting C so that the interior of C lies entirely on one side of the line.

A line or line segment ℓ containing a point P of C is a *normal* if and only if ℓ is orthogonal to some support line of C through P.

Let C_1 be an arc lying in the upper half-plane and having its endpoints at (-a, 0) and (a, 0) on the x-axis. C_1 is a *convex arc* if and only if, together with its chord from (-a, 0) to (a, 0), it forms a closed convex curve.

The *moment* of C_1 about the x-axis, denoted by $I(C_1)$, is given by

(2) $$I(C_1) = \int_{C_1} y\,ds.$$

The integralgeometric definition of the area A(S) of a surface S in Euclidean space E_3 is as follows: let dG be the usual integralgeometric density for the set of

Received November 26, 1963.