THE RADIUS OF UNIVALENCE OF CERTAIN ANALYTIC FUNCTIONS

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In two recent papers, MacGregor presented the following results [3, pp. 527-519] and [4, pp. 522-523].

Suppose that $f(z) = z + a_2 z^2 + \cdots$ *and* $g(z) = z + b_2 z^2 \cdots$ *are analytic in the unit disc* $D = \{z \mid |z| < 1\}$.

(a) *If* g *is univalent in* D *and if* $\Re[f(z)/g(z)] > 0$ *in* D, *then* f *is univalent in the disc* $|z| < 1/5$.

(b) *If* g *is univalent and star-like in* D *and if* $\Re[f(z)/g(z)] > 0$ *in* D, *then* f *is univalent and star-like in the disk* $|z| < (2 - \sqrt{3})$.

(c) *If* g *is univalent in* D *and if* $|f(z)/g(z) - 1| < 1$ *in* D, *then* f *is univalent in the disc* $|z| < 1/3$.

(d) *If* g *is univalent and star-like in* D *and if* $|f(z)/g(z) - 1| < 1$ *in* D, *then* f *is univalent and star-like in* D.

As MacGregor points out, the radii in (b), (c) and (d) are the best possible, while that in (a) is not known to be the best possible radius.

In this note we present two theorems that improve the results stated above.

THEOREM 1. *Suppose that* $f(z) = z + a_2 z^2 + \cdots$ *and* $g(z) = z + b_2 z^2 + \cdots$ *are analytic in the unit disc* D, *and suppose that* $\Re[f(z)/g(z)] > 0$ *in* D. *If* g *is univalent in* D, *then* f *is univalent and star-like in the disc* $|z| < (2 - \sqrt{3})$. *This result is sharp.*

Our proof of Theorem 1 depends on the following result, which has independent interest.

LEMMA. *Let* $g(z) = z + b_2 z^2 + \cdots$ *be analytic and univalent in the unit disc* D. *Then the inequality*

$$\Re \frac{zg'(z)}{g(z)} \geq \frac{1 - |z|}{1 + |z|} \tag{1}$$

holds for $|z| < \tanh \frac{1}{2} = 0.46212 \cdots$. *The bound* (1) *is sharp for each* z; *it is attained by a rotation of the Koebe function* $k(z) = z/(1 - z)^2$.

Proof. It is known that for each z in D the complex number $u = \log [zg'(z)/g(z)]$ lies in the closed disc $\overline{K}(\rho)$ whose center is at the origin and whose radius is $\rho = \log [(1 + |z|)/(1 - |z|)]$ [2, p. 113]. Now the function $w = e^u$ maps $\overline{K}(\rho)$ onto a convex region whenever the inequality

$$\Re \left(1 + u\frac{w''(u)}{w'(u)} \right) = \Re(1 + u) \geq 0$$

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