SOME INVARIANTS OF p-GROUPS

R. A. Beaumont and R. S. Pierce

1. INTRODUCTION

The purpose of this paper is to define and study a certain system of invariants of primary abelian groups without elements of infinite height. The invariants take the form of ideals in the Boolean algebra $P(\omega)$ of all subsets of the set ω of finite ordinals. It is natural to consider the existence and uniqueness of p-groups with a given associated invariant. The main results of the paper are concerned with the existence problems.

All of the groups considered in this paper are assumed to be p-primary abelian groups, where p is some fixed prime. Most of the notation is taken from [3] and [6]. If $x \in G$, the height $h_G(x)$ is defined to be the maximum k such that $x \in p^k G$ if this maximum exists, and $h_G(x) = \infty$ if $x \in p^k G$ for all k. The subgroup of all $x \in G$ with $h_G(x) = \infty$ is denoted by G^1. A subgroup H of G is *pure in* G if $h_H(x) = h_G(x)$ for all $x \in H$. We shall denote by $f_G(k)$ the kth Ulm invariant of G:

$$f_G(k) = \dim (p^k G \cap G[p] / p^{k+1} G \cap G[p]).$$

It is convenient to adjoin the definition $f_G(\infty) = \dim (G^1 \cap G[p])$.

We consider cardinal and ordinal numbers in the sense of von Neumann; that is, an ordinal number is a set, namely, the set of all smaller ordinals. Cardinal numbers are ordinal numbers that are not equivalent to any smaller ordinal. The cardinal number of the set X is denoted by $|X|$. The set of all subsets (the power-set) of X is represented by P(X). The symbol ω denotes the first infinite ordinal, that is, the set of all finite ordinals. The letter $\mathfrak{c}$ represents the cardinal number of the continuum. The symbols $\subset$ and $\supset$ denote inclusion in the wide sense. Finally, it is convenient to write ω^+ to denote the set $\omega \cup \{\infty\}$.

2. THE INVARIANTS

We first define a general class of invariants, then focus our attention on one of particular interest.

2.1 *Definition.* $I(G) = \{k \in \omega^+ \mid f_G(k) \neq 0\}$.

Evidently, $f_G(k) \neq 0$ if and only if $p^k G \cap G[p] / p^{k+1} G \cap G[p] \neq 0$, that is, there is an $x \in G[p]$ such that $h_G(x) = k$. Moreover, $f_G(\infty) \neq 0$ if and only if there is a nonzero $x \in G[p]$ with $h_G(x) = \infty$. Hence:

2.2 LEMMA. $I(G) = \{h_G(x) \mid x \in G[p],\ x \neq 0\}$.

2.3 COROLLARY. *If* H *is a pure subgroup of* G, *then* $I(H) \subset I(G)$.

2.4 COROLLARY. *If* H *and* K *are pure subgroups of* G *such that* $I(H) \cap I(K) = \emptyset$, *then* $H \cap K = 0$.

Received July 12, 1963.
This work was supported by National Science Foundation grant No. GP-809.