SOME OBSERVATIONS ON QUASICOHESIVE SETS

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1. In this note we shall derive a few results concerning *quasicohesive* sets of natural numbers. One or two of these results would be of considerable interest if they could be obtained for sets with recursively enumerable (r. e.) complements. Even in their present broad form, they lead to some moderately interesting facts, as we shall see in Section 5.

The results contained in Section 4 answer questions that were raised partly by J. S. Ullian and partly by myself.

After the original version of this paper was submitted for publication, I learned that Prof. Hartley Rogers has proved, in a draft of Chapter 12 of his forthcoming book on recursive function theory, a result closely related to Theorem 1. In fact, Rogers' proof establishes just exactly the first of the two assertions comprising the statement of Theorem 1. (His construction does not furnish the additional information that K_2 may be any cohesive subset of a cosimple set.)

D. A. Martin, in a letter of Sept. 18, 1963, has communicated to me the following theorem: there are nonhyperhyperimmune, r-cohesive infinite number sets. This result implies both Theorem 2 and Theorem 3 of the present paper. Martin's construction is substantially different from the one I have used for Theorems 2 and 3.

2. As usual, 'W_j' denotes the j-th term in some fixed uniform enumeration of the class of all r. e. sets of natural numbers. We use 'N' to denote the set of all natural numbers. A subset α of N is called *cohesive* (the term is due to Ullian) provided α is infinite and, for each j, either $\alpha \cap W_j$ or $\alpha \cap \overline{W}_j$ is finite. (In general, we use '$\bar{\alpha}$' as notation for N - α, α any subset of N.) By a *quasicohesive* set of natural numbers we mean one that is a finite union of cohesive sets. We say that a subset β of N *splits* a set $\alpha \subset$ N (the symbol '$\subset$' indicates inclusion in the wide sense) if $\alpha \cap \beta$ and $\alpha \cap \bar{\beta}$ are both infinite. Finally, we call a subset α of N r-*cohesive* provided α is infinite and is not split by any recursive set. Sometimes, r-cohesive sets are referred to as being "recursively indecomposable." [An infinite set α of natural numbers is said to be *decomposable* if and only if there exist disjoint sets W_i, W_j such that (1) both $\alpha \cap W_i$ and $\alpha \cap W_j$ are infinite, and (2) $\alpha = (\alpha \cap W_i) \cup (\alpha \cap W_j)$.]

3. The lemmas listed in this section form the basis of the proofs to be presented in Section 4.

LEMMA 1. *If* S *is a nonempty, countable collection of infinite subsets of* N *with the property that if* s_i, $s_j \in$ S *then also* $s_i \cap s_j \in$ S, *then there exists an infinite set* β *such that* $\beta \subset \bigcup$S *and* β - s_i *is finite for all* $s_i \in$ S.

Lemma 1 was proved by Dekker in [1].

LEMMA 2. *There exists an r. e. set with a cohesive complement.*

Lemma 2 was proved by Friedberg in [3].

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