AN ELEMENTARY PROOF OF KATĚTOV'S THEOREM CONCERNING Q-SPACES

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We recall here that a completely regular (Hausdorff) space X is called a Q-space [1] provided that every homomorphism ϕ of the ring C(X) of all real continuous functions defined on X into the ring of real numbers which does not vanish identically on C(X) is of the form

$$\phi(f) = f(p_0) \qquad \text{for all } f \text{ in } C(X),$$

where p_0 is a fixed point of X. Q-spaces can be characterized in a purely topological manner; for instance, it is shown in [5] that a completely regular space X is a Q-space if and only if it satisfies the following condition.

(Q): *for every point* p_0 *from* $\beta X \setminus X$, *there exists a function* $f\colon \beta X \to I$ *such that* $f(p_0) = 0$ *and* $f(p) > 0$ *for* p *in* X.

I denotes here the closed unit interval [0, 1]; $f\colon \beta X \to I$ means that f is a continuous function which maps βX into I.

In [3] Katětov has proved the following theorem.

THEOREM. *If* X *is a paracompact space and every closed discrete subspace of* X *is a* Q-*space, then* X *is also a* Q-*space.*

(This theorem is a particular case of Shirota's result [6]: *if a space* X *admits a complete uniformity and every closed discrete subspace of* X *is a* Q-*space, then* X *is a* Q-*space.* Indeed, every paracompact space admits a complete uniformity.)

We shall give here another, more elementary proof of Katětov's theorem. We begin with the following remarks.

Clearly, the problem whether a discrete space is a Q-space depends only upon the cardinality of the space. Moreover, if R is a discrete space and R_0 is an arbitrary subspace of R, then $\overline{R_0} = \beta R_0$, where $\overline{R_0}$ denotes the closure of R_0 in βR. Hence, using the condition (Q), one can easily infer that if R is a Q-space, then R_0 is also. Therefore we can state:

(i) *if* R_1 *and* R_2 *are discrete spaces with* $\overline{\overline{R}}_1 \leq \overline{\overline{R}}_2$, *and* R_2 *is a* Q-*space, then* R_1 *is also a* Q-*space.*

Denote as m_0 the least cardinal such that the discrete space of the cardinality m_0 is not a Q-space. (It can be shown [2], that m_0 is the so-called *first measurable cardinal;* this fact, however, will not be used in our reasonings. In particular, the non-existence of such cardinal would only simplify the proof.) According to (i), it follows that:

(ii) *a discrete space* R *is a* Q-*space if and only if* $\overline{\overline{R}} < m_0$.

Notice that if $\{F_\xi\colon \xi \in \Xi\}$ is a discrete system of subsets of a space (we recall here that a system of subsets of a space is said to be *discrete* provided that each point of the space has a neighbourhood which intersects at most one member of the

Received April 25, 1963.