RELATIVE INVERSES IN BAER *-SEMIGROUPS

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1. INTRODUCTION

In this note we give some results on relative inverses in Baer *-semigroups which seem to have interesting consequences when interpreted in terms of the Moore-Penrose generalized inverse for (square) matrices. In [7] Penrose shows that if A is an n-by-m matrix over the real or the complex field, then there exists a unique m-by-n matrix A^+, called the *generalized inverse* of A, such that $A = AA^+A$, $A^+ = A^+AA^+$, $(AA^+)^* = AA^+$, and $(A^+A)^* = A^+A$ (where A^* is the transposed conjugate of a matrix A). Our methods apply only to square matrices, but it seems plausible that they can be extended to rectangular matrices by some device such as the adjunction of rows or columns of zeros.

We begin with several definitions. An *involution semigroup* is a semigroup S together with a mapping $*: S \to S$ such that (i) $(xy)^* = y^*x^*$ and (ii) $x^{**} = x$ for all $x, y \in S$. A *projection* in such an S is an element $e \in S$ with $e = e^2 = e^*$. We denote by $P = P(S)$ the partially ordered set of all projections in S, the partial order being defined by the condition that $e \leq f$ if and only if $e = ef$ $(e, f \in P)$.

A *Baer *-semigroup* is an involution semigroup S with a two sided zero 0 having the following property: For each element $s \in S$ there exists a projection $s' \in P$ such that $\{x \in S \mid sx = 0\} = s'S$. It is clear that the projection s' is uniquely determined by s, since two principal right ideals generated by projections in an involution semigroup S are equal if and only if the projections are equal. The notion of a Baer *-semigroup was introduced (in a slightly more general form) in [1].

If a is an element of the Baer *-semigroup S, then we say that a is **-regular* in S if there exists a (necessarily unique) element a^+ in S such that $a = aa^+a$, $a^+ = a^+aa^+$, $aa^+ = (a^*)''$, and $a^+a = a''$. (Actually, the concept of *-regularity will be given a slightly different, but equivalent, working definition in Section 4.) The element a^+ will be called the *relative inverse* of a in S; it clearly specializes to the Moore-Penrose generalized inverse of a if S is the Baer *-semigroup of all n-by-n matrices over the real or the complex field.

If a and b are *-regular elements of the Baer *-semigroup S, it is natural to ask for conditions which will guarantee that ab is also *-regular in S. In Section 7 we show that *-regularity of the product $a''(b^*)''$ is a necessary and sufficient condition for *-regularity of the product ab; and we obtain, in this case, the formula

$$(ab)^+ = (ab)''\, b^+ (a''(b^*)'')^+ a^+ (b^* a^*)''$$

for the relative inverse of ab. This formula reduces the problem of computing the relative inverse of the product of two (square) matrices to the problem of computing the relative inverse of the product of two projection matrices.

If a is an invertible element in the Baer *-semigroup S, then clearly $a^{-1} = a^+$. If both a and b are invertible, then $(ab)^{-1} = b^{-1}a^{-1}$. Hence it is natural to ask for conditions equivalent to $(ab)^+ = b^+a^+$ if a and b are not necessarily invertible, but

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