FACTORING OF SECOND-ORDER DIFFERENCE EQUATIONS WITH PERIODIC COEFFICIENTS

E. Dubinsky

We consider the following difference equation:

$$f(x+i) - p(x)f(x) - f(x-i) = 0 \qquad (i = \sqrt{-1}) \tag{1}$$

under the hypothesis that there exists an integer n for which

$$p(x+ni) = p(x).$$

Equation (1) is completely general, for if

$$a(x)g(x+i) + b(x)g(x) + c(x)g(x-i) = 0,$$

then, setting

$$g(x) = r(x)f(x),$$

where

$$a(x)r(x+i) = -c(x)r(x-i),$$

we obtain (1) with

$$p(x) = \frac{b(x)r(x)}{a(x)r(x+i)}.$$

A continued fraction solution of (1) is given in [1], but no discussion of convergence is given there, and in practice to prove convergence may be quite difficult. The periodic coefficient case is discussed by Fort [2], under the restriction that the values of f are required only at integral multiples of i. He shows the existence of a second order equation with constant coefficients, some of whose solutions comprise all the solutions of (1). His result is useful in considering the asymptotic behavior of solutions.

We shall show that the problem of solving (1) can be reduced to that of solving certain first order linear difference equations whose coefficients we give explicitly in terms of p and n. There are two cases. In the first case, we give two homogeneous first order linear difference equations with the property that any pair of solutions of these two equations are solutions of (1) and are linearly independent. (Throughout this paper, linear independence is with respect to the ring of functions that have period i.) In the second case, these two equations turn out to be identical and a second, independent, solution is obtained as a solution of a nonhomogeneous, first order, linear difference equation with periodic coefficients whose right-hand side is the solution of a homogeneous equation.

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