INNER AUTOMORPHISMS OF GROUPS IN TOPOLOGICAL ALGEBRAS

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1. INTRODUCTION. Let B be a complex Banach algebra with an identity e, and let G be the multiplicative group of all regular elements of B. This group plays an important role in the theory of Banach algebras; for expositions and further references see [3] and [5]. Here we investigate the group $\mathfrak{J}$ of all inner automorphisms of G. The group $\mathfrak{J}$ can equally well be considered as the group of all inner automorphisms of the algebra B inasmuch as, given u, v $\in$ G, the equality $uxu^{-1} = vxv^{-1}$ holds for all x $\in$ G if and only if it holds for all x $\in$ B. The group $\mathfrak{J}$ is, of course, isomorphic to the group G/Z, where Z is the center of G, and is trivial if B is commutative; we are concerned only with algebras which are not commutative.

The quotient group formed by a group modulo its center can, as is well known, readily have a nontrivial center. However, we show in Theorem 2.3 that, for all semi-simple Banach algebras B (or, more generally, for any normed Q-algebra B whose center is semi-simple), the group G/Z has only the identity in its center. (In the special case where B is the algebra of all matrices of degree n over the complex field K, G is the general linear group GL(n, K), Z is the set of nonzero scalar multiples of the identity, and G/Z is isomorphic to the projective group in n - 1 dimensions over K; see, for example, [1, p. 297]. In this case Theorem 2.3 states that the projective group has a trivial center.) That G/Z has only the identity in its center is true in spite of the fact that, for all such B which are not commutative, the power of G/Z is at least that of the continuum. The latter property holds in a more general setting (Theorem 2.6), but there are incomplete real normed algebras that are not commutative and for which G/Z is trivial.

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2. ON THE GROUP G/Z. As in [4], we call a topological ring A a *Q-ring* if the set of quasi-regular elements of A is open; A is a Q-ring if and only if there is a neighborhood of zero consisting entirely of quasi-regular elements [4, p. 154]. By [3, p. 695], any modular maximal right (left) ideal of A is closed. Any Banach algebra is a Q-algebra [4, p. 155]. Suppose that A is a Q-algebra over the reals with identity e. Then to each x $\in$ A there corresponds a real number $b \neq 0$ such that $e + bx \in G$. It follows that $Z = C \cap G$, where C is the center of A.

For an element x in an algebra A over the real or complex numbers, we denote its spectrum by Sp(x). If x lies in a subalgebra A_1 and we wish to consider its spectrum when x is considered as an element of A_1, we denote this set by $Sp(x) | A_1)$.

Normed Q-algebras have been investigated in [6].

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