ONE-PARAMETER SUBGROUPS IN SEMIGROUPS IN THE PLANE

J. G. Horne, Jr.

1. INTRODUCTION

Let S be a topological semigroup (with identity) that is embedded in the plane E. Suppose that P is a one-parameter subgroup in S containing the identity (that is, suppose P is algebraically and topologically isomorphic to the multiplicative group of positive real numbers). In this note, our interest centers on two questions: (1) What is the nature of an orbit Px for $x \in S$ if Px is neither a point nor a simple closed curve? (2) What is the nature of the boundary of P? We show, without any restriction on S, that the answer to the first question is that Px is homeomorphic to P. Assuming that S is closed, we show that the boundary of P (if not empty) is either a point (which is a zero for P) or the circle group. If $S = E$, then that stage in the proof of each result which makes use of the main lemma can be made somewhat simpler, because if the local cross section theorem is applied to the plane, it yields a section which is an arc (see [6]). Since an orbit can cross a section at most once, it becomes an easy matter to construct the simple closed curves needed in order to show that if an orbit enters a certain region and if it must get out, then it must "cross in the opposite direction." Alternately, if it is known that a certain orbit *can not* cross in the opposite direction, then one end of it must be bounded, and one can find an idempotent. We make these ideas precise and use them in the proof of Theorem 1 and (by way of Lemma 1) in the proof of Theorem 2. They are precisely the ideas used in the proof of Theorem 3.7 of [7] and Lemma 2.1 of [6]. Perhaps the most striking illustration of the difference between the case $S = E$ and the general case is obtained by comparison of the figure in [7] with our figure.

So far, the only applications we have of the case $S \neq E$ are slightly technical. One yields Corollary 1.1. For another, observe that the nature of the boundary of P seems to rule out any "reasonable" semigroup structure for a set T which is the union of the curve $y = \sin 1/x$ $(0 < x < 1)$ with an open arc in its boundary. For the closure of T, such a structure has long been ruled out. However, even in the case $S = E$, our results appear to have gone unnoticed hitherto.

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2. THE MAIN LEMMA

The first lemma is purely topological. It is likely to seem obvious, but the proof is not entirely trivial. As was indicated in the Introduction, this lemma makes it possible to carry out arguments like those in [6] and [7], where the local cross section theorem does not yield enough information. All of the results of this paper depend on it; and as we intimate in the final remark, the results of Mostert in [6] on the nature of the boundary of G can be derived from it.

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