THE GEOMETRY OF EXTREMAL QUASICONFORMAL MAPPINGS

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1. In this paper we consider Teichmüller mappings between orientable C^2 surfaces immersed (without singularities) in E^3. We investigate the geometric consequences of the assumption that certain standard differential geometric correspondences are Teichmüller mappings. Since conformal mappings are the simplest Teichmüller mappings, it is not surprising that our results tend to generalize known facts about the conformal mapping of surfaces in E^3.

In order to describe the extremal problems which lend importance to Teichmüller mappings, and in order to define these mappings in an appropriate setting, we begin with a brief review of some material from the theory of quasiconformal mapping. A thorough explanation of the subject matter outlined in Section 2 can be found in [1] or [2]; but for a quick reading of the theorems proved here, it is enough to assume that Teichmüller mappings satisfy the conclusion of Lemma 1 in Section 3.

2. Quasiconformal mappings of plane domains may be defined as follows. Let $\mathscr{R}$ denote a *topological rectangle* in E^2, that is, a closed Jordan region with four distinguished boundary points. Then there exists a unique value m ($m \geq 1$) such that some homeomorphism, *conformal in the interior of* $\mathscr{R}$, carries $\mathscr{R}$ onto the classical rectangle

$$0 \leq x \leq m, \qquad 0 \leq y \leq 1$$

in E^2 in such a way that the four distinguished boundary points are mapped onto the four vertices of the classical rectangle. The value m is called the *modulus* of $\mathscr{R}$ (notation: $m = \operatorname{mod} \mathscr{R}$).

An arbitrary (sense-preserving) homeomorphism $w: D \to E^2$ of a plane domain D carries each topological rectangle $\mathscr{R} \subset D$ onto a topological rectangle $w(\mathscr{R})$. Such a w is k-*quasiconformal* if and only if

$$\text{l.u.b.}\ \frac{\operatorname{mod} w(\mathscr{R})}{\operatorname{mod} \mathscr{R}} = K = \frac{1+k}{1-k} < \infty,$$

where $\mathscr{R}$ ranges over the class of all topological rectangles $\mathscr{R} \subset D$.

The constant K is called the *maximal dilatation* of w, a name it receives because of the special case in which w is in C^1 and has a positive Jacobian. For here, w carries an infinitesimal circle at any p in D onto an infinitesimal ellipse at w(p) whose major and minor axes are in the ratio $K(p) \geq 1$. In this context, K(p) is called the *dilatation of* w *at* p, and w is k-quasiconformal (by our previous definition) if and only if

$$\sup_{p \in D} K(p) = K = \frac{1+k}{1-k} < \infty .$$

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