A CHARACTERIZATION OF THE ANALYTIC OPERATOR AMONG THE LOEWNER-BENSON OPERATORS

R. E. Lewkowicz

1. INTRODUCTION

C. Loewner [1] considered integral operators of the type

$$(1) \qquad y(t) = -\int_0^{2\pi} K(s)\,x(t - s)\,ds = -\int_0^{2\pi} K(t - s)\,x(s)\,ds,$$

where $K(t)$ is L-integrable on the interval $[0, 2\pi]$ and $x(t)$ ranges over the continuous 2π-periodic functions. He gave necessary and sufficient conditions that such operators generate only curves $\{x(t), y(t)\}$ of non-negative circulation, that is, curves whose index relative to any point not on them is non-negative. His conditions are

(a) $K(t)$ is (possibly after a change in its values on a set of measure zero) analytic in the open interval $(0, 2\pi)$ and

(b) $K'(t)$ can be represented, in the interval $(0, 2\pi)$, by a Laplace-Stieltjes integral

$$K'(t) = \int_{-\infty}^{\infty} e^{-rt}\,d\mu(r),$$

where $\mu(t)$ is a non-decreasing function.

D. C. Benson [2, 3] extended Loewner's result to include the case where $K(t)$ is not necessarily L-integrable on the closed interval $[0, 2\pi]$ but is such that the Cauchy Principal Value $P\int_0^{2\pi} K(t)\,dt$ exists. For a certain class of continuous periodic functions, he showed that, in order that the operator (1) (with the integral understood as a Cauchy Principal Value) generate only curves of non-negative circulation, it is again necessary and sufficient that conditions (a) and (b) hold.

Among the kernels which fall into Benson's class is the kernel $K(t) = -\cot t/2$. This kernel corresponds to what we have called the analytic operator—the operator that relates, on the boundary of the unit disk, the real and imaginary parts of a function continuous on the closed disk and analytic on the interior. The analytic operator

$$y(t) = P\int_0^{2\pi} \cot\frac{s}{2}\; x(t - s)\,ds$$

has the property that if $m(t)$ is a continuous mapping of the line onto itself induced by a one-to-one conformal map of the closed disk onto itself, then

Received June 1, 1961.