THE DISTRIBUTION OF PRIME ENDS

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1. INTRODUCTION

Carathéodory defined and classified the prime ends of a simply connected domain, and he raised the question how the four kinds of prime ends of a domain can be distributed (see [1], especially the last paragraph of the Introduction). Weniaminoff was the first to show that the space of prime ends of a simply connected domain can not contain an arc of prime ends of the second kind [10]. The question was further pursued by Collingwood, who proved that the prime ends of the second and fourth kinds (that is, the prime ends with subsidiary points) constitute a set of first category [2, p. 349]. For the special case where only prime ends of the first and second kinds are present, Lohwater and I showed that the prime ends of the second kind form a set of type F_σ (and of first category), and that this result is "best possible" [6, p. 8]. Recently, I solved the problem for the special case where only prime ends of the first three kinds are present [7, p. 50]. Now I shall give the solution for the general case.

2. DEFINITIONS

It will be convenient to have at hand both Carathéodory's classification of prime ends and Lindelöf's function-theoretic interpretation of the classification. Let P be a prime end of a simply connected domain B, and I(P) its impression, that is, the set of boundary points of B which is naturally associated with the equivalence class constituting the prime end P. A point p in I(P) is a *principal point* of P provided the prime end has a chain of crosscuts converging to p (in the Euclidean metric); it is a *subsidiary point* of P if it is not a principal point of P. A prime end is of the *first, second, third* or *fourth kind* according as its impression consists of

(i) only one point (necessarily a principal point),

(ii) one principal point and some subsidiary points,

(iii) more than one principal point and no subsidiary points,

(iv) more than one principal point and some subsidiary points.

For a more complete discussion and an illustration, see [7, p. 47].

If the function $\phi(\zeta)$ maps the unit disk $|\zeta| < 1$ conformally onto a simply connected domain B, it induces a one-to-one correspondence between the set of points $\zeta = e^{i\theta}$ $(0 \leq \theta < 2\pi)$ and the set of prime ends of B [1, p. 350]. Also, the radial cluster set of ϕ at $e^{i\theta}$ coincides with the set of principal points of the corresponding prime end [5, p. 28]. Therefore the prime end corresponding to $e^{i\theta}$ is of the first or second kind if and only if the radial limit of ϕ at $e^{i\theta}$ exists, and it is of the first or third kind if and only if the radial cluster set of ϕ at $e^{i\theta}$ is identical with the complete cluster set of $e^{i\theta}$.

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