ON MAPS WITH NONNEGATIVE JACOBIAN

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The purpose of this note is to prove the following theorem.

THEOREM 1. *Let* M *and* N *be two oriented* n*-dimensional differentiable manifolds with* M *compact and* N *connected. Let* f *be a differentiable map of* M *into* N *whose Jacobian* J(f) *is nonnegative. Then either* $J(f) \equiv 0$, *or* N *is compact,* f *is onto, and* f *has positive degree on each component of* M *on which* $J(f) \not\equiv 0$.

Remark. Since the two manifolds are oriented, that is, since we have chosen a fixed orientation for both, the sign of the Jacobian at any point of M is well defined. The special case of this theorem where M and N are surfaces is treated in [1].

In the course of the proof of this theorem we shall give a proof in modern terminology, and without the use of triangulations, of some classical results about degrees of maps. The results below can also be stated and proved for relative manifolds without any serious modification.

We thank Mr. W. Pohl for suggesting this problem to us.

1. ORIENTATION AND DEGREE

Let M be a connected n-manifold (not necessarily differentiable). If $x \in M$ and if $U \subset M$ is a cell containing x, then $H_n(M, M - x) \approx H_n(U, U - x)$ by excision, and $H_n(U, U - x) \approx Z$. A generator of $H_n(M, M - x)$ is called a local orientation at x. If x and y are two points contained in the same open n-cell V, there is a canonical isomorphism $\phi^V_{x,y}$, depending only on V, from $H_n(M, M - x)$ to $H_n(M, M - y)$, defined as follows. By excision, the inclusion maps $(M, M - V) \subset (M, M - x)$ and $(M, M - V) \subset (M, M - y)$ induce isomorphisms on the corresponding H_n. Taking the composition of these in the obvious way gives $\phi^V_{x,y}$. A manifold M is called orientable if we can choose a generator μ_x for every $H_n(M, M - x)$ in a consistent manner; that is, if for any V, x, and y,

$$\phi^V_{x,y}\,\mu_x = \mu_y\,. \tag{1}$$

LEMMA 1. *Let* $\mathfrak{V}$ *be a covering of* M *by open cells. A necessary and sufficient condition for* M *to be orientable is that* (1) *hold for all* $V \in \mathfrak{V}$.

The necessity of the condition is trivial. As to its sufficiency, it is clear that if $x, y \in V' \subset V$, then

$$\phi^{V'}_{x,y} = \phi^V_{x,y}\,. \tag{2}$$

Thus if (1) holds for V, it holds for V'. In particular, (1) holds for all sufficiently small V. Now let U be any cell containing x and y. Choose an arc γ containing x

Received May 19, 1959.

This research was sponsored by the Office of Ordnance Research, U.S. Army, under Contract DA-11-022-ORD-2911.