THE SETS OF AMBIGUOUS POINTS OF FUNCTIONS OF BOUNDED CHARACTERISTIC

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If $f(z)$ is a complex-valued function defined in the unit disk D, a point $P = e^{i\theta}$ is called an *ambiguous point* of $f(z)$ provided there exists a simple closed Jordan curve Γ, interior to D except for the point P, such that $f(z)$ approaches two different limits as $z \to P$ along Γ from opposite sides. Bagemihl [1, Theorem 2] has shown that even if the function $f(z)$ is not required to be continuous in D, its set of ambiguous points is at most denumerable. On the other hand, every denumerable set on the unit circle C is contained in the set of ambiguous points of some function which is regular and of bounded characteristic in D (Bagemihl and Seidel [2]). It is our purpose to prove that in the last statement the words "is contained in" can be replaced by "coincides with."

THEOREM. *Let* E *be a denumerable set on* C. *Then there exists a function* $f(z)$ *which is regular and of bounded characteristic in* D *and whose set of ambiguous points coincides with* E.

Let $E = \{z_n\}$, let $\{a_n\}$ be a decreasing sequence of positive constants ($\Sigma a_n < 1$), and let D_n denote the disk of diameter a_n which is internally tangent to C at z_n. Clearly the a_n can be chosen in such a way that no two of the disks D_n intersect. We consider the function

$$f(z) = g(z)/h(z) = g(z)\,k(z),$$

where

$$g(z) = \exp\left(- \sum a_n^2/\sqrt{1 - z/z_n}\right),$$

$$k(z) = \exp \sum a_n^2/(1 - z/z_n).$$

In the formula for $g(z)$, the symbol $\sqrt{1 - z/z_n}$ represents that branch of the corresponding function which has the value 1 at $z = 0$. That $f(z)$ is regular in D is seen directly from the formulas. Also, since $\Re(1 - z/z_n) > 0$ throughout D, the two quantities $\Re \log g(z)$ and $\Re \log h(z)$ are negative throughout D, and therefore $f(z)$ is of bounded characteristic.

Next we observe that the function $w = u + iv = a_n^2/(1 - z/z_n)$ maps the disk D_n onto the half-plane $u > a_n$, and the complementary set $D - D_n$ onto the strip $a_n^2/2 < u \leq a_n$. From this it follows that, for each m, the inequality

$$\Re \log k(z) < \sum a_n < 1$$

is satisfied on the boundary C_m of D_m (taken relative to D), and, more generally, that $|k(z)| < e$ in the set $D^* = D - \bigcup D_n$. Since $\Re \log g(z) \to -\infty$ as $z \to z_m$ along

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