LUSIN'S THEOREM ON AREAS OF CONFORMAL MAPS

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1. INTRODUCTION

If the function $f(z)$ is of class H_2, then there exists a convex domain D whose boundary is tangent from the interior to the unit circle C at $z = 1$ and which has the following property: the area $A(f, D, \theta)$ of the Riemann surface onto which the function $w = f(e^{i\theta}z)$ maps the domain D is an integrable function of θ. We shall refer to this proposition as Lusin's theorem. Lusin actually claimed a little less; he did not assert that $A(f, D, \theta)$ is an integrable function, merely that it is finite for almost all θ; but his proof [2, pp. 139-149] clearly establishes the stronger proposition.

In Section 2 we give a brief proof of Lusin's theorem. Conceptually, our proof is identical with that of Lusin; technically it is somewhat simpler, because of a profitable reversal of order in an iterated integral. In addition, our slight modification yields a converse of Lusin's theorem.

In a second proof of the theorem, we construct the domain D in terms of the Taylor coefficients of the function f. We also show that even the weak form of Lusin's theorem becomes false if the hypothesis of bounded mean square modulus is replaced by the hypothesis of slow growth of the maximum modulus; also that the domain D in Lusin's theorem can not be chosen independently of the function f.

Lusin [2, p. 151] conjectured that the property of the function f in his theorem is essentially a local rather than a global property. This is indeed the case: Let f be meromorphic in $|z| < 1$; then, for almost all points $e^{i\theta}$ at which the cluster set of f for nontangential approach is not identical with the entire plane, there exists a convex domain $D(\theta)$, touching C at $e^{i\theta}$, such that the image of $D(\theta)$ under f has finite area.

In Section 3, we consider the exceptional set relative to Lusin's theorem, that is, the set of points $e^{i\theta}$ for which $A(f, D, \theta) = \infty$ for every convex domain D in $|z| < 1$ which touches C at $z = 1$. Lusin stated [2, p. 142] that even if f is continuous in $|z| \leq 1$, the exceptional set need not be empty; we illustrate this statement with a simple example.

On the other hand, if the Taylor series of f converges absolutely on C, then the exceptional set is empty; in fact, here the quantity $A(f, D, \theta)$ is a continuous function of θ, for some convex domain D touching the unit circle. This result heightens the remarkable character of the theorems which assert that certain Taylor series converging absolutely on C map C onto a Peano curve (see Salem and Zygmund [4] and Schaeffer [5]).

In Section 4, we waive the requirement that D be convex and that its boundary have a tangent; of the function f we require only that it be holomorphic in $|z| < 1$. From the fact that, for $f(z) = \Sigma a_n z^n$ and $0 \leq r < 1$,

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