CONVERGENCE PROPERTIES OF SEQUENCES OF LINEAR FRACTIONAL TRANSFORMATIONS

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1. INTRODUCTION

We are concerned with two aspects of the convergence behavior of arbitrary sequences of transformations

$$T_n = T_n(z) = \frac{A_n z + B}{C_n z + D} \qquad (n = 1, 2, \cdots). \tag{1}$$

One aspect is the nature of the point set where such a sequence converges; the other is the character of the limit function, $T(z)$, on that point set. While the two aspects of the problem are not quite independent, the connection between them is surprisingly slight, and they can be treated almost separately.

In order to make the further discussion precise, we mention here some conventions. No cases of interest will be lost through the assumption that all of the transformations T_n are nonsingular; we shall therefore assume, throughout, that $A_n D_n - B_n C_n \neq 0$. Also, the subject naturally demands that the transformations be regarded as mappings of the extended plane onto itself; therefore a sequence of points will be called divergent only if it has at least two limit points in the extended plane. A point set E will be called a set of convergence provided some sequence (1) converges everywhere on E and nowhere outside of E. The complement of a set of convergence will be called a set of divergence, or an SD, for short. In other words, the statement "E is an SD" shall have the meaning: "there exists a sequence (1) which diverges everywhere on E and converges everywhere outside of E."

It happens that the limit function of (1) is very simple, throughout the set of convergence, regardless of how the set of convergence is constituted. In fact, only finitely many essentially different situations can occur. Therefore we shall first treat the properties of the limit function (Section 2), and then we shall study the more difficult problem of finding the point sets which are sets of divergence (Sections 3 and 4). This latter problem is not yet completely solved. The continuity of the linear fractional transformations implies that every SD is a set of type $G_{\delta\sigma}$ (see [1], p. 273). On the other hand we shall prove, for example, that every set of type G_δ (but not every set of type F_σ) is an SD, and that not every SD is a set of type G_δ. We shall also show that it is not possible to characterize the SD's in purely topological terms. The essential reason for this state of affairs is the fact that the level curves of the function $1/(z - h)$ are concentric circles.

The present study originated from Thron's investigations on the convergence behavior of continued fractions. If $t_n(z) = a_n/(b_n + z)$ and

$$T_n(z) = t_1(t_2 \cdots (t_n(z)) \cdots),$$

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