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Universal Complexes and the Generic Structure of Free Resolutions

Alexandre B. Tchernev

Introduction

An important aspect of modern commutative algebra is the study of the structure of finite free resolutions. The first significant result in this direction goes back to Hilbert [22]; in its most general form, due to Burch [11], it describes the structure of free resolutions of length 2 whose component in degree 0 is a free module of rank 1. This theorem was generalized by Buchsbaum and Eisenbud [10], who obtained structure theorems for arbitrary finite free resolutions. The question of whether these are the "best possible" structure theorems was one of the topics of Hochster's influential CBMS lectures [23]. Hochster's approach to this problem is to describe a generic resolution of a given type from which all other resolutions of the same type are obtained by base change.

To be specific, let R be a commutative algebra over a (fixed) base ring $\Bbbk$, and let

$$\mathbf{F} = 0 \to R^{b_n} \xrightarrow{X^{(n)}} R^{b_{n-1}} \to \cdots \to R^{b_1} \xrightarrow{X^{(1)}} R^{b_0} \to 0 \qquad (\dagger)$$

be a complex, where $X^{(k)} = (x_{ij}^{(k)}) \neq 0$ is the matrix of the kth differential in the standard bases of R^{b_k} and $R^{b_{k-1}}$, $k = 1, \dots, n$. Hochster calls the pair $(R, \mathbf{F})$ a *universal pair* if $\mathbf{F}$ is acyclic and if, for each commutative $\Bbbk$-algebra S and each free resolution

$$\mathbf{G} = 0 \to S^{b_n} \xrightarrow{Z^{(n)}} S^{b_{n-1}} \to \cdots \to S^{b_1} \xrightarrow{Z^{(1)}} S^{b_0} \to 0, \qquad (\ddagger)$$

there exists a unique $\Bbbk$-algebra homomorphism $u\colon R \to S$ such that $u(x_{ij}^{(k)}) = z_{ij}^{(k)}$; thus $\mathbf{G} = \mathbf{F} \otimes_R S$. When it exists, a universal pair $(R, \mathbf{F})$ is determined up to isomorphism by the sequence of its *Betti numbers* $\boldsymbol{b} = (b_0, \dots, b_n)$; we call R the *universal ring of type* $\boldsymbol{b}$ *over* $\Bbbk$, and $\mathbf{F}$ the *universal resolution of type* $\boldsymbol{b}$ *over* $\Bbbk$.

A main step in Hochster's program is to establish the values of $b_0, \dots, b_n$ for which a universal pair exists. Hochster [23] (when $\Bbbk$ is either the ring of integers $\mathbb{Z}$, or a field) and later Bruns [5] (in general) show that, when $n \leq 2$, a necessary and sufficient condition for existence is that the "expected ranks" $r_k = \sum_{s=k}^{n} (-1)^{s-k} b_s$ satisfy $r_0 \geq 0$ and $r_k \geq 1$ for $1 \leq k \leq n$. When $n \geq 3$, Bruns [5] shows that universal pairs do not exist, regardless of the choice of the numbers $b_0, \dots, b_n$.

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