Michigan Math. J. 66 (2017), 389–419

Zero Distribution of Random Sparse Polynomials

Turgay Bayraktar

Abstract. We study the asymptotic zero distribution of random Laurent polynomials whose supports are contained in dilates of a fixed integral polytope P as their degree grow. We consider a large class of probability distributions including those induced from i.i.d. random coefficients whose distribution law has bounded density with logarithmically decaying tails and moderate measures defined over the space of Laurent polynomials. We obtain a quantitative localized version of the Bernstein–Kouchnirenko theorem.

1. Introduction

Recall that the Newton polytope of a Laurent polynomial $f(z_1,\dots,z_m) \in \mathbb{C}[z_1^{\pm1},\dots,z_m^{\pm1}]$ is the convex hull (in $\mathbb{R}^m$) of the exponents of monomials in $f(z)$. It is well known that for a system $(f_1,\dots,f_m)$ of Laurent polynomials in general position, the common zeros is a discrete set in $(\mathbb{C}^*)^m := (\mathbb{C}\setminus\{0\})^m$ and that the number of simultaneous zeros of such a system is given by the mixed volume of Newton polytopes of f_i [Ber75; Kou76]. In this work, we study the asymptotic behavior of zeros of the systems of random Laurent polynomials with prescribed Newton polytope as their degree grow. More precisely, we consider Laurent polynomials whose supports are contained in dilates NP for a fixed integral polytope $P \subset \mathbb{R}^m$ with nonempty interior. Random Laurent polynomials with independent identically distributed (i.i.d.) coefficients whose distribution law is absolutely continuous with respect to Lebesgue measure and has logarithmically decaying tails arise as a particular case. In particular, standard real and complex Gaussians are among the examples of such distributions. In another direction, moderate measures defined on the space of Laurent polynomials also fall into framework of this paper.

Computation of simultaneous zeros of deterministic and Gaussian systems of sparse polynomials has been studied by various authors (see e.g. [HS95; Roj96; MR04; DGS14]) by using mostly methods of algebraic and toric geometry. In this work, we employ methods of pluripotential theory (cf. [SZ04; DS06a; BS07; CM15; BL15; Bay16]), which is extensively used in the dynamical study of holomorphic maps (see [FS95] and references therein). Along the way, we develop a pluripotential theory for plurisubharmonic (psh for short) functions that are dominated by the support function of P (up to a constant) in logarithmic coordinates on $(\mathbb{C}^*)^m$. We remark that the class of psh functions that we work with is a generalization of the Lelong class, which corresponds here to the particular case $P = \Sigma$

Received January 28, 2016. Revision received November 27, 2016.