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Sofic Profile and Computability of Cremona Groups

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0. Synopsis

In this paper, we show that Cremona groups are sofic. We actually introduce a quantitative notion of soficity, called sofic profile, and show that the group of birational transformations of a d-dimensional variety has sofic profile at most polynomial of degree d. We also observe that finitely generated subgroups of the Cremona group have a solvable word problem. This provides examples of finitely generated groups with no embeddings into any Cremona group, answering a question of S. Cantat.

1. Introduction

Let K be a field. The *Cremona group* $\mathrm{Cr}_d(K)$ of K in dimension d is defined as the group of birational transformations of the d-dimensional K-affine space. It can also be described as the group of K-automorphisms of the field of rational functions $K(t_1, \dots, t_d)$.

We are far from a global understanding of finitely generated subgroups of Cremona groups. They include, notably, linear groups (since we have an obvious inclusion $\mathrm{GL}_d(K) \subset \mathrm{Cr}_d(K)$) as well as examples of groups that are not linear over any field [CeD]. On the other hand, very few restrictions are known about these groups. In the case of $d = 2$, and sometimes assuming that K has characteristic 0, there has been a lot of recent progress including [Be; BeB; B1; B2; B3; Do; DoI1; DoI2]; see notably the survey [Se2] about finite subgroups and [B2; BD1; BD2; Ca1; D] for other subgroups. For $d = 3$ there is much less information currently known; in this direction, see [Pr1; Pr2; PrSh] concerning finite subgroups. For greater d, very little information is known; interesting methods have recently been developed in [Ca2].

We here provide the following.

Theorem 1.1. *The Cremona group* $\mathrm{Cr}_d(K)$ *is sofic for all* d *and all fields* K. *More generally, for any absolutely irreducible variety* X *over a field* K, *the group of birational transformations* $\mathrm{Bir}_K(X)$ *is sofic.*

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