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One-Dimensional Metric Foliations on Compact Lie Groups

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A *k-dimensional metric foliation* on a Riemannian manifold M is a decomposition of M into locally equidistant (immersed) k-dimensional submanifolds called *leaves.* The homogeneous foliations—that is, foliations whose leaves are locally orbits of an isometric group action—are the primary source of metric foliations. As is well known, not all metric foliations are homogeneous. Nevertheless, it would be interesting to determine the spaces on which the homogeneity property does hold. One has complete results in this direction if the leaves are one-dimensional and the sectional curvature of the space is constant. Indeed, all one-dimensional metric foliations on spaces of constant nonnegative sectional curvature are homogeneous, whereas spaces of negative sectional curvature admit an abundance of nonhomogeneous one-dimensional metric foliations [1]. However, less is known if the constant curvature assumption is dropped. Among the few manifolds with nonconstant curvature on which it is known that the homogeneity property holds for one-dimensional foliations, we can mention $S^2 \times \mathbb{R}$ [2] and the Heisenberg group [3]. Our main purpose is to show that the class of manifolds just described also includes the compact Lie groups equipped with bi-invariant metrics and their quotients by finite groups acting freely and isometrically. It is also interesting to remark that the result is not valid without compactness. A counterexample on $\mathrm{SL}_2(\mathbb{R})$ is given in [7].

1. Introduction

In this section we introduce some notation and recall some well-known facts about compact Lie groups. Our main concern is the form of the Jacobi vector fields. It is this form that will play a key role in our investigations of one-dimensional metric foliations.

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, and consider an inner product on $\mathfrak{g}$ such that $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is orthogonal for each $g \in G$. If T is a maximal torus, then we have the following orthogonal decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = V_0 \oplus \sum_{r=1}^{M} V_r,$$

where V_0 is the Lie algebra of T and each V_r is a two-dimensional subspace, $1 \le r \le M$. Moreover, for each $x \in V_0$, $\mathrm{ad}_x = [x, \cdot] : \mathfrak{g} \to \mathfrak{g}$ acts trivially on V_0 and

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