Michigan Math. J. 52 (2004)

Formulas for the Dimensions of Some Affine Deligne–Lusztig Varieties

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1. Introduction

Let F be $\mathbb{F}_q((t))$ with ring of integers $\mathcal{O}_F$, and let G be a split connected reductive group over F. Let L be the completion of the maximal unramified extension of F, $\bar{\mathbb{F}}_q((t))$. Let σ be the Frobenius automorphism of L over F. Let $\mathcal{B}_n$ be the affine building for $G(E)$ where E/F is the unramified extension of degree n in L (so $E = \mathbb{F}_{q^n}((t))$), and let $\mathcal{B}_\infty$ be the affine building for $G(L)$. Let T be a split torus in G, let $B = UT$ be a Borel subgroup, and let I be an Iwahori in $G(L)$ containing $T(\mathcal{O}_L)$, where $\mathcal{O}_L$ is the ring of integers of L. Let A_M and C_M be the correspondingly specified apartment and alcove, which we assume are in $\mathcal{B}_1$; we will call these the *main apartment* and the *main alcove,* respectively. We assume that C_M is in the positive Weyl chamber in A_M specified by B. Let $P \supseteq I$ be a parahoric subgroup of $G(L)$. If $b \in G(L)$ then the σ-conjugacy class of b is $\{x^{-1}b\sigma(x) : x \in G(L)\}$. Let $\tilde{W} = N(L)/T(\mathcal{O}_L)$ be the extended affine Weyl group, and let $\tilde{W}_P = N(L) \cap P/T(\mathcal{O}_L)$. Here N is the normalizer of T.

If $\tilde{w} \in \tilde{W}$, then we define (after Rapoport [12] and Kottwitz) the *generalized affine Deligne–Lusztig variety* $X^P_{\tilde{w}}(b\sigma) = \{x \in G(L)/P : \mathrm{inv}_P(x, b\sigma(x)) = \tilde{w}\}$. Here $\mathrm{inv}_P : G(L)/P \times G(L)/P \to P\backslash G(L)/P = \tilde{W}_P\backslash\tilde{W}/\tilde{W}_P$ is the relative position map associated to P. Rapoport [12] asked which pairs $(b, \tilde{w})$ give rise to non-empty sets and, for these pairs, what is $\dim(X^P_{\tilde{w}}(b\sigma))$. Kottwitz and Rapoport [9; 12] answered the emptiness/non-emptiness part of this question for $P = K$, the maximal bounded subgroup of $G(L)$ associated to some special vertex v_M of C_M.

In Section 3 we consider the case $G = \mathrm{SL}_3$ with $b = 1$ and $P = I$. Complete results on emptiness/non-emptiness and dimension are shown for this case in Figure 5. In Section 4 we consider $G = \mathrm{Sp}_4$, again with $b = 1$ and $P = I$. Emptiness/non-emptiness results and dimension results are shown in Figure 10. The case $G = \mathrm{SL}_2$ ($b = 1$, $P = I$) can be handled using an even simpler version of the same methods.

Rapoport showed in [12, Prop. 4.2] that, for general G, $X^K_{\tilde{w}}(\sigma)$ is non-empty for any $\tilde{w}$ corresponding to a dominant cocharacter in the coroot lattice. This is also shown in some special cases in [9]. Rapoport [13] conjectured a specific formula for the dimension of the $X^K_{\tilde{w}}(\sigma)$. The knowledge of the $X^I_{\tilde{w}}(\sigma)$ mentioned in the

Received March 17, 2003. Revision received September 8, 2003.
The author was partially supported by NSF Grant no. DEB 9981552.