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Correspondences between $K3$ Surfaces

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(with an Appendix by Igor Dolgachev)

1. Introduction

In this paper we study the existence of correspondences between $K3$ surfaces $X(k,m,n)$ with $k,m,n \in \mathbf{N}$, Picard number 17, and transcendental lattices $T(k,m,n) \cong U(k) \oplus U(m) \oplus \langle -2n \rangle$. In a fundamental paper, Mukai [Mu] showed that correspondences between $K3$ surfaces exist if the transcendental lattices are Hodge isometric over $\mathbf{Q}$. This construction holds if the Picard number of the surfaces is greater than or equal to 11. Nikulin [N2] later improved this result, obtaining the lower bound 5 for the Picard number.

The aim of our work is to realize examples of $K3$ surfaces with transcendental lattice that are not Hodge isometric but such that a correspondence between them already exists. This in particular implies the existence of an algebraic cycle on the middle cohomology of the product of two surfaces arbitrarily chosen in the constructed family.

In Sections 2 and 3 we recall some basic notions and results on lattices and correspondences. In Section 4 we consider a generic genus-2 curve and we show the existence of a correspondence between the Jacobian of the curve and a $K3$ surface with isomorphic transcendental lattice. Since this construction involves a second $K3$ surface whose transcendental lattice has quadratic form multiplied by 2, in Sections 5 and 6 we generalize this first example. First, we construct $K3$ surfaces "twisting" each direct summand of the transcendental lattice of the Jacobian by natural numbers. Then we find correspondences between them using both Mukai's theorem and Shioda–Inose structures that translate the problem into a problem of looking for isogenies between abelian varieties.

In this way we prove in Theorem 6.3 that all the $K3$ surfaces $X(k,m,n)$ are in correspondence to each other. Finally, in Theorem 6.5 we show the existence of a correspondence between a general $K3$ surface of Picard rank 17 and a Kummer surface of the same rank having transcendental lattices that are $\mathbf{Q}$-Hodge isomorphic.

In the Appendix, Igor Dolgachev realizes a geometric correspondence between the $K3$ surfaces of Section 4.

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