

# Extremal Elliptic Surfaces and Infinitesimal Torelli

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## 1. Introduction

An extremal elliptic surface over  $\mathbb{C}$  is an elliptic surface such that the rank of the Néron–Severi group equals  $h^{1,1}$  and there are finitely many sections.

Their main application, until now, is in the classification of singular fibers on certain elliptic surfaces: Once a configuration of singular fibers on an extremal elliptic surface is known, one can construct from this configuration many other configurations of singular fibers on elliptic surfaces, where the genus of the base curve and the geometric and arithmetic genus of the surface remain fixed. In [S] these operations are called elementary transformations and are, a priori, valid only for  $K3$  surfaces. Actually, all elementary transformations are combinations of twisting and deformations of the  $J$ -map (terminology from [M1, VIII.2; M2]) and hence are valid for any elliptic surface.

The classification of singular fibers on a rational elliptic surface was given more than ten years ago (see [M2; OSh; P]). Recently there has been given a classification of all singular fibers of elliptic  $K3$  surfaces with a section (see [S]). From the classification of configurations of singular fibers on rational surfaces (see [M2]) and  $K3$  surfaces (see [S]), we know that any configuration can be obtained from an extremal configuration using elementary transformations.

In this paper we give a complete classification of extremal elliptic surfaces with constant  $j$ -invariant (Theorem 3.3). We use this classification to prove the following.

**THEOREM 1.1.** *Let  $\pi : X \rightarrow \mathbb{P}^1$  be an elliptic surface without multiple fibers. Assume that  $p_g(X) > 1$ . Then  $X$  does not satisfy infinitesimal Torelli if and only if  $j(\pi)$  is constant and  $\pi$  is extremal.*

Kii [K, Thm. 2] proved infinitesimal Torelli for elliptic surfaces without multiple fibers and nonconstant  $j$ -invariant. Saitō [Sa] proved in a different way infinitesimal Torelli for elliptic surfaces without multiple fibers and  $j$ -invariant different from 0 and 1728.

For elliptic surfaces with nonconstant  $j$ -invariant, we will give the following structure theorem.