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Poles Near the Origin Produce Lower Bounds for Coefficients of Meromorphic Univalent Functions

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1. Introduction

Let D denote the open unit disc. In this paper we consider functions f meromorphic and univalent in D that have a simple pole at the point $p \in D \setminus \{0\}$. For $r \in (0,1)$ fixed, let U_r denote the class of all such functions f, $|p| = r$, that are normalized by $f(0) = 0$ and $f'(0) = 1$. Hence, any function $f \in U_r$ has an expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n, \quad |z| < r,$$

and $f(p) = \infty$.

As usual, we denote by S the class of functions g holomorphic and univalent in D with Taylor coefficients $a_n(g) = g^{(n)}(0)/n!$, where $a_0(g) = 0$ and $a_1(g) = 1$. By de Branges's famous proof [4] of the validity of the Bieberbach conjecture, it is known that the domain of variability of $a_n(g)$ for $g \in S$ and $n \ge 2$ is the whole disc defined by

$$|a_n(g)| \le n \tag{1}$$

and that equality in (1) is attained if and only if

$$g(z) = \kappa_d(z) := \frac{z}{(1 - \bar{d}z)^2}, \quad |d| = 1. \tag{2}$$

In [9] Goodman conjectured that, for $f \in U_r$ with $r \in (0,1)$, the inequalities

$$|a_n(f)| \le \frac{1}{r^{n-1}} \sum_{k=0}^{n-1} r^{2k}, \quad n \ge 2, \tag{3}$$

are valid. Jenkins [12] proved that (3) is true for $a_N(f)$ if (1) is valid for $n = 2, \dots, N$. Hence (3) holds for all $n \ge 2$. From Jenkins's proof it is also evident that equality in (3) is attained if and only if

$$f(z) = \kappa_p(z) := \frac{z}{(1 - \bar{p}z)(1 - z/p)}, \quad |p| = r. \tag{4}$$

Concerning the inverse functions g^{-1} of $g \in S$, a classical theorem of Löwner [19] indicates that, for the Taylor coefficients $A_n(g^{-1})$, the inequalities

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