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Lattice Points inside Random Ellipsoids

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1. Introduction

Let

$$N_a(t) = \#\{t\Omega_a \cap \mathbb{Z}^d\}, \tag{0.1}$$

where

$$\Omega_a = \left\{(a_1^{-\frac{1}{2}}x_1, a_2^{-\frac{1}{2}}x_2, \dots, a_d^{-\frac{1}{2}}x_d) : x \in \Omega\right\} \tag{0.2}$$

with $\frac{1}{2} \le a_j \le 2$ and where Ω is the unit ball.

Let

$$N_a(t) = t^d|\Omega_a| + E_a(t). \tag{0.3}$$

A classical result due to Landau states that

$$|E_a(t)| \lesssim t^{d-2+\frac{2}{d+1}}; \tag{0.4}$$

here and throughout the paper, $A \lesssim B$ means that there exists a positive constant C such that $A \le CB$. Similarly, $A \lessapprox B$, with a parameter t, means that given $\delta > 0$ there exists a $C_\delta > 0$ such that $A \le C_\delta t^\delta B$.

A number of improvements over (0.4) have been obtained over the years in two and three dimensions. The best-known result in three dimensions (to the best of our knowledge) is $|E_a(t)| \lessapprox t^{\frac{21}{16}}$ proved by Heath-Brown [HB], improving on an earlier breakthrough due to Vinogradov [V]. It is proved by Szegö that

$$\left|E_{1,1,1}(t) - \frac{4\pi}{3}t^3\right| \gtrsim t\log(t). \tag{0.5}$$

In two dimensions, the best-known result is $|E_a(t)| \lessapprox t^{\frac{46}{73}}$ due to Huxley [Hu]. A classical result due to Hardy says that

$$|E_{1,1}(t) - \pi t^2| \gtrsim t^{\frac{1}{2}} \log^{\frac{1}{2}}(t). \tag{0.6}$$

Thus it is reasonable to conjecture that the estimate

$$|E_a(t)| \lessapprox t^{\frac{d-1}{2}} \tag{0.7}$$

holds in $\mathbb{R}^2$ and $\mathbb{R}^3$.

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