The Convex Hull of the Interpolating Blaschke Products

Michael D. O'Neill

1. Introduction and Notation

In the sequel we prove that if a Blaschke product B is continuous in the closed unit disk except on a closed set $E \subset \mathbb{T}$ of measure zero, then B is contained in $9K$, where K denotes the closed convex hull of the interpolating Blaschke products. Moreover, we show that a generic Blaschke product is contained in $27K$. By the well-known theorem of Marshall, this implies that the unit ball of H^∞ is contained in $27K$. The proofs employ a technical result, given in Section 3, which may be of some independent interest. The results in the paper improve earlier work in [8], [4], and [10]. We refer to these papers and to [3] for further background on the questions treated here.

We shall employ the following notation:

$\mathbb{D}$ open unit disk, $\mathbb{D} \equiv \{ z \in \mathbb{C} : |z| < 1 \}$;

$\mathbb{T}$ unit circle, $\mathbb{T} \equiv \partial\mathbb{D}$;

H^∞ the space of bounded analytic functions in $\mathbb{D}$;

L^∞ the space of essentially bounded functions on $\mathbb{T}$;

$P_z(w)$ the Poisson kernel in $\mathbb{D}$, $P_z(w) = (1-|z|^2)/|1-\bar{w}z|^2$;

$\rho(z,w)$ the "pseudo hyperbolic distance" between z and w in $\mathbb{D}$,

$$\rho(z,w) \equiv \left| \frac{z-w}{1-\bar{w}z} \right|;$$

$d(z,w)$ the hyperbolic distance between z and w in $\mathbb{D}$,

$$d(z,w) \equiv \log \frac{1+\rho(z,w)}{1-\rho(z,w)};$$

$d(X,Y)$ for $X \subset \mathbb{D}$ and $Y \subset \mathbb{D}$,

$$d(X,Y) \equiv \inf\{ d(x,y) : x \in X,\ y \in Y \};$$

A_δ^ε an annulus of thickness 2ε about the circle $\{|z| = \delta\}$,

$$A_\delta^\varepsilon \equiv \{ z \in \mathbb{D} : \delta - \varepsilon \le |z| \le \delta + \varepsilon \};$$

I a half-open subinterval of $\mathbb{T}$,

$$I = \{ e^{i\theta} \in \mathbb{T} : \theta_1 \le \theta < \theta_2 \} \quad \text{for some } 0 \le \theta_1 < \theta_2 \le 2\pi;$$

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