ON A THEOREM OF GIFFEN

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0. Introduction. This paper is based on some ideas of Giffen concerning the Karoubi conjecture. Let R be a ring with unit and an involution. Let $\mathrm{GL}(R) = \varinjlim \mathrm{GL}_n(R)$ be the infinite general linear group and ${}_\epsilon O(R) = \varinjlim {}_\epsilon O_{n,n}(R)$ be the infinite orthogonal group where $\epsilon = \pm 1$ and ${}_\epsilon O_{n,n}(R)$ is the group of automorphisms of R^{2n} preserving the ϵ-hermitian pairing

$$\langle x, y \rangle = x \left[\begin{array}{c|c} 0 & I_n \\ \hline \epsilon I_n & 0 \end{array}\right] \bar{y}^t.$$

Applying the plus-construction to the classifying spaces $B\mathrm{GL}(R)$ and $B{}_\epsilon O(R)$ of these groups we obtain the classifying spaces of algebraic K-theory and hermitian K-theory, respectively.

During the early development of K-theory, Karoubi studied the relation between these two theories. The natural inclusion ${}_\epsilon O_{n,n}(R) \to \mathrm{GL}_{2n}(R)$ and the hyperbolic map $\mathrm{GL}_n(R) \to {}_\epsilon O_{n,n}(R)$ which takes

$$g \mapsto \left[\begin{array}{c|c} g & 0 \\ \hline 0 & (\bar{g}^t)^{-1} \end{array}\right],$$

induce mappings $B{}_\epsilon O(R)^+ \to B\mathrm{GL}(R)^+$ and $B\mathrm{GL}(R)^+ \to B{}_\epsilon O(R)^+$. Motivated by periodicity in topological K-theory, Karoubi [7] conjectured that the homotopy fibers ${}_\epsilon \mathcal{V}(R)$ and ${}_\epsilon \mathcal{U}(R)$ of these maps are related by a homotopy equivalence, $\Omega\, {}_\epsilon \mathcal{U}(R) \simeq {}_{-\epsilon}\mathcal{V}(R)$. This conjecture was proven to be true by Karoubi and Loday ([9], [10]) under the assumption that 2 is invertible in R.

More recently, Giffen attempted to reinterpret this conjecture in a categorical framework. For this, he introduced a category ${}_\epsilon W(R)$ intended as a model for a delooping of ${}_\epsilon \mathcal{U}(R)$. These ideas have never appeared in print, but have significant implications in light of recent work by the current authors. The main purpose of this paper, therefore, is to describe the category ${}_\epsilon W(R)$ and to prove that there exists a homotopy fibration of infinite loop spaces

$$K_0(R) \times B\mathrm{GL}(R)^+ \to K_0^H(R) \times B{}_\epsilon O(R)^+ \to |{}_\epsilon W(R)| \tag{0.1}$$

as predicted by Giffen.

Our own interest in this problem stems from the study of compactifications of moduli spaces. In [5], we construct maps from the Satake compactification of Siegel space, $\mathfrak{S}_n^*$, to certain subspaces $|{}_{-1}W_n(\mathbf{Z})|$ of $|{}_{-1}W(\mathbf{Z})|$, and prove that these maps induce isomorphisms on rational cohomology. Using a special case of the fibration (0.1) (proved in the appendix of [5]) and computations of Borel, we are able to determine this cohomology in degrees $<n$. Recent work of the authors

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