GENERALIZED HOMOLOGY THEORIES ON COMPACT METRIC SPACES

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1. INTRODUCTION

This paper is devoted to developing useful and tractable homology theories on the category $\mathscr{CM}$ of based compact metrizable spaces, and doing this, moreover, within the context of classical algebraic topology. In the introduction we explain why this is desirable and we then state our main results. Let $\mathscr{A}$ be the category of abelian groups.

DEFINITION 1.1 [28]. A *Steenrod homology theory* h_* on $\mathscr{CM}$ is a sequence of covariant, homotopy-invariant functors $h_n: \mathscr{CM} \to \mathscr{A}$ such that the following axioms hold for all n and for all X in $\mathscr{CM}$:

Exactness. If A is a closed subset of X then the sequence

$$h_n(A) \to h_n(X) \to h_n(X/A)$$

is exact.

Suspension. There is a natural equivalence $h_n(X) \xrightarrow{\sigma} h_{n+1}(SX)$.

Strong Wedge. Suppose X_j is in $\mathscr{CM}$, $j = 1, 2, \cdots,$. Then the natural map

$$h_n(\varprojlim_k (X_1 \vee \cdots \vee X_k)) \to \prod_j h_n(X_j)$$

is an isomorphism.

Classical (ordinary) Steenrod homology theory is denoted ${}^sH_*$. It was invented by Steenrod [40] and axiomatized by Milnor [34]. The theory ${}^sH_*$ is very well-behaved on $\mathscr{CM}$. Steenrod showed that it is related to Čech homology by the sequence

$$(1.2) \qquad 0 \to \varprojlim{}^1 H_{n+1}(X_j) \to {}^sH_n(X) \to \check{H}_n(X) \to 0\,,$$

where $X = \varprojlim X_j$ and the X_j are finite complexes. This is a special case of the $\lim^1$ sequence of Milnor [34]

$$(1.3) \qquad 0 \to \varprojlim{}^1 h_{n+1}(X_j) \to h_n(X) \to \varprojlim h_n(X_j) \to 0\,,$$

which holds for any Steenrod homology theory.

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