LIFTING OF OPERATORS THAT COMMUTE WITH SHIFTS

J. G. W. Carswell and C. F. Schubert

1. INTRODUCTION

Throughout this paper, all Hilbert spaces are assumed to be complex, all operators are assumed to be linear and bounded, and all subspaces are assumed to be closed.

If S is a unilateral shift on a Hilbert space $\mathscr{H}$, $\mathscr{M}$ is a subspace of $\mathscr{H}$ that is invariant under S^*, and T is an operator on $\mathscr{M}$ commuting with S^*, then there exists an operator T_1 on $\mathscr{H}$, also commuting with S^*, that satisfies the condition $\|T_1\| = \|T\|$, and the restriction $T_1 \mid \mathscr{M}$ coincides with T. This particular result has been proved in different ways by several authors [2], [4], [7]. Here we shall consider a related lifting question that is in the nature of an adjoint problem.

If $\mathscr{M}$ is a subspace of $\mathscr{H}$ that is invariant under S and T is an operator on $\mathscr{M}$ commuting with S, does there exist an operator T_1 on $\mathscr{H}$ that also commutes with S and is such that $T_1 \mid \mathscr{M} = T$? Unlike the lifting problem for co-isometries above, this problem has no solution unless a subsidiary condition is satisfied. For each integer $k \geq 1$, the operator $P_k = I - S^k S^{*k}$ is an orthogonal projection on $\mathscr{H}$. If T_1 is an operator on $\mathscr{H}$ commuting with S, then $P_k T_1 P_k = P_k T_1$. Thus, for all $u \in \mathscr{H}$ and all $k \geq 1$, we have

$$\|P_k T_1 u\| = \|P_k T_1 P_k u\| \leq \|T_1\| \, \|P_k u\| .$$

In particular, if $T = T_1 \mid \mathscr{M}$, then for all $u \in \mathscr{M}$, T must satisfy the inequality $\|P_k T u\| \leq \|T_1\| \, \|P_k u\|$. Thus, if an operator T on $\mathscr{M}$ has an extension to $\mathscr{H}$ that commutes with S, then the value

$$\alpha = \sup_{k \geq 1} \sup_{u \in \mathscr{M}} \frac{\|P_k T u\|}{\|P_k u\|} \tag{1.1}$$

must necessarily be finite. Note that by (1.1) $\alpha \geq \|T\|$, while by the remarks above, $\alpha \leq \|T_1\|$ for any extension T_1. L. B. Page [5] conjectured that the condition (1.1) is also sufficient to ensure that T has an extension T_1 to all of $\mathscr{H}$ such that $T_1 S = S T_1$; he conjectured further that if T_1 is an extension of minimal norm, then $\|T_1\| = \alpha$, and he was able to prove this conjecture in several special cases. For shifts of finite multiplicity, the conjecture was proved in [6] by means of a Hardy space model. The object of this paper is to prove this conjecture in the general form given here.

We shall prove our results by reducing the problem to the lifting theorem for co-isometries.

Received December 24, 1974.
This work was supported by National Research Council Grant A-7352.

Michigan Math. J. 22 (1975).