PERIODIC SOLUTIONS OF ANALYTIC FUNCTIONAL DIFFERENTIAL EQUATIONS ARE ANALYTIC

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In 1955, E. M. Wright [11] studied the nonlinear differential-difference equation

$$x'(t) = -\alpha x(t-1)(1+x(t)) \qquad (\alpha > 0),$$

and he proved among many other results that each real solution $x(t)$ of this equation that is defined, continuously differentiable, and bounded on the real axis has a complex analytic extension on the strip $|\Im(t)| < (\alpha e^{\alpha})^{-1}$. Wright's method is based on the observations that $x(t)$ is necessarily infinitely differentiable and that repeated differentiation of the defining equation leads to estimates on $|x^{(n)}(t)|$. The estimates on $|x^{(n)}(t)|$ imply the analyticity of $x(t)$. The same technique works for some other equations, for example, the equation

$$x'(t) = -\left(\sum_{j=1}^{n} \alpha_j x(t-\tau_j)\right)(1+x(t)),$$

where α_j and τ_j denote positive constants; but for more general equations it is by no means obvious how to obtain the appropriate estimates on $|x^{(n)}(t)|$.

In 1962, G. S. Jones [7] proved that if $\alpha > \pi/2$, then the equation

$$x'(t) = -\alpha x(t-1)(1+x(t))$$

has a nontrivial periodic solution; by Wright's work, this solution is necessarily analytic on a strip. Jones also proved in [8] that if $\alpha > \pi/2$, the equation $x'(t) = -\alpha x(t-1)(1-x^2(t))$ has a nontrivial periodic solution that is analytic on a strip. These results and a number of other special cases led him to ask the following question in [8]: *If* η *is a real-valued function of bounded variation, and if* α, a, b, *and* h *are constants, is each real-valued periodic solution* $x(t)$ *of the equation*

$$x'(t) = \left(-\alpha \int_{-h}^{0} x(t+\theta)\, d\eta(\theta)\right)(1 + ax(t) + bx^2(t))$$

analytic on a neighborhood of $\mathbb{R}$ *in the complex plane?*

The answer is yes. We shall prove a much more general theorem. The proof is surprisingly straightforward. The idea is simply to apply some abstract methods that reduce the question to one of analytic solutions of an ordinary differential equation with values in a complex Banach space. We have enough information about the ordinary differential equation to prove the result.

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