ISOLATED SUBGROUPS

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A subgroup H of a group G is *isolated* provided its conjugates are strictly disjoint; explicitly, H is isolated provided $xHx^{-1} \cap H = \{1\}$ whenever $x \in G$ and $x \notin H$. Isolated subgroups seem to have been used only in Frobenius's theorem for finite groups and its developments [4]. Our aim is to consider the effect on the structure of a (possibly infinite) group G of its supply of isolated subgroups. At one extreme, G has no isolated subgroups except $\{1\}$ and G (G is I-*simple*); at the other extreme, G admits a nontrivial partition by isolated subgroups (G is *multic*). Most well-known classes of groups are *monic*, that is, nonmultic (Sections 1, 2); however, we obtain several noteworthy classes of multic groups. Our interest in these questions arose from geometry, and in Section 5 we show that the isolated subgroups of the fundamental group of a Riemannian manifold M are closely related to the curvature of M. Finally, in Section 6 we discuss finite and infinite Frobenius groups.

Our late colleague Theodore Motzkin participated in the beginning investigations of this paper. We consider him a coauthor, even though the completed paper could not have his customary meticulous scrutiny.

1. TOTAL GROUPS

1.1. LEMMA. (1) *The intersection of an arbitrary collection of isolated subgroups of* G *is isolated.*

(2) *If* A *is an isolated subgroup of* B, *and* B *is an isolated subgroup of* C, *then* A *is isolated in* C.

(3) *If* I *is an isolated subgroup of* G *and* H *is a subgroup of* G, *then* $I \cap H$ *is isolated in* H.

(4) *If* I *is isolated in* G *and* $x \in G$, *then* $x^n \in I \setminus \{1\}$ *implies* $x \in I$.

(5) *No proper isolated subgroup of* G *contains a nontrivial normal subgroup of* G.

By the first of these properties, if S is a subset of a group G, we may define I_S to be the smallest isolated subgroup of G containing S. In particular, for each $x \in G$ we have the isolated subgroup I_x. An element $x \in G$ is *total* if $I_x = G$.

We now distinguish some classes of groups that have successively richer supplies of isolated subgroups.

1.2. *Definition.* (1) G is I-*simple* if $\{1\}$ and G are the only isolated subgroups of G.

(2) G is *total* if it contains a total element.

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