UNIVERSALLY COMMUTATABLE OPERATORS ARE SCALARS

Paul R. Chernoff

1. INTRODUCTION

Let A and B be finite-dimensional linear operators that generate one-parameter groups $P_t = e^{tA}$ and $Q_t = e^{tB}$. Then the groups generated by A + B and AB - BA can be expressed by means of the well-known Lie product formulas

$$\lim_{n \to \infty} (P_{t/n} Q_{t/n})^n = \exp t(A + B), \tag{1}$$

$$\lim_{n \to \infty} (P_{-\sqrt{t/n}}\, Q_{-\sqrt{t/n}}\, P_{\sqrt{t/n}}\, Q_{\sqrt{t-n}})^n = \exp t(AB - BA). \tag{2}$$

An infinite-dimensional version of (1) was proved by H. F. Trotter [9]. It states that (1) is valid if P_t and Q_t are (C_0) contraction semigroups on a Banach space such that the closure $[A + B]^-$ of the sum of their generators itself generates a (C_0) semigroup R_t. The right side of (1) is to be interpreted as R_t, and the limit is in the strong operator topology. In [1], the present author proved a rather general theorem that includes Trotter's. J. A. Goldstein [5] and E. Nelson [8, Theorem 8.7] have used this result to prove infinite-dimensional versions of the commutator formula (2).

The limits in (1) and (2) may exist even when the hypotheses of [5], [8], and [9] are not satisfied. By our general theory (see [2], [3]) the limits must be semigroups, if they exist at all. If they are (C_0) semigroups, we denote their generators by $A +_L B$ and $[A, B]_L$. The subscript L refers to a generalized Lie operation. These generalized operations can be quite pathological. A detailed study of generalized addition of self-adjoint operators is contained in [3]; a number of examples concerning both addition and commutation can be found in [6]. In particular, we showed in [3] that only *bounded* self-adjoint operators A can be added--by the Lie process or by any other reasonable process—to every self-adjoint operator B. In fact, if A is not bounded, then one can construct a B such that the symmetric operator A + B, defined on $\mathscr{D}(A) \cap \mathscr{D}(B)$, has no self-adjoint extensions.

Goldstein [6] has conjectured that an analogous situation holds for commutators. Let $\mathscr{H}$ be an infinite-dimensional Hilbert space. Call a self-adjoint operator A *universally commutatable in the classical sense* if for all self-adjoint B the operator AB - BA, defined on $\mathscr{D}(AB) \cap \mathscr{D}(BA)$, is essentially skew-adjoint; call A *universally commutatable in the Lie sense* if $[A, B]_L$ exists for all self-adjoint B. The only operators that are obviously universally commutatable (in either case) are the scalar multiples of the identity. As we shall show, there are no other operators universally commutatable in the classical sense or the Lie sense, at least if the definition of the latter is strengthened in a technical way.

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