HARDY CLASSES AND RANGES OF FUNCTIONS

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I. INTRODUCTION

1. Let D be a region (that is, a connected, nonempty, open set) in the complex plane $\mathscr{C}$. Following M. Parreau [6] and W. Rudin [8], for each positive real number p, we let $H_p(D)$ denote the collection of functions f, analytic on D, for which $|f|^p$ has a harmonic majorant. (In the case where D is the unit disk, $H_p(D)$ as just defined coincides with the usual Hardy class H_p.) We let $H_0(D)$ denote the collection of analytic functions on D. *For each fixed function* $f \in H_0(D)$, *we seek to determine, by studying* f(D), *the numbers* p *for which* $f \in H_p(D)$.

One of the first results in this direction is due to Smirnov [7, p. 64]. He showed that if f is analytic on Δ, where $\Delta = \{|z| < 1\}$, and has positive real part, then $f \in H_p(\Delta)$ $(0 < p < 1)$. It is an easy step to go from Smirnov's Theorem to the result that $f \in H_p(\Delta)$ $(0 < p < \pi/\alpha)$ if $f(\Delta)$ is contained in a sector whose angular opening is α $(0 < \alpha \leq 2\pi)$. This was pointed out by G. T. Cargo [2], who also proved the following results for a function $f \in H_0(\Delta)$:

(1) If $f(\Delta) \subseteq \Omega \subsetneq \mathscr{C}$, where Ω is simply-connected, then $f \in H_p(\Delta)$ $(0 < p < 1/2)$.

(2) If $f(\Delta)$ is contained in an infinite strip, then $f \in H_p(\Delta)$, for all positive numbers p.

Cargo proved these last two results using the principle of subordination. Thus, the existing results are limited to the case where $f(\Delta) \subseteq \Omega \subsetneq \mathscr{C}$ and Ω is simply-connected.

We begin by introducing the *Hardy number* $h(\Omega)$ of a region $\Omega \subseteq \mathscr{C}$ (Chapter II). The Hardy number $h(\Omega)$ has the property that if $f \in H_0(D)$, $f(D) \subseteq \overline{\Omega}$, and $h(\Omega) > 0$, then $f \in H_p(D)$ $(0 < p < h(\Omega))$. Therefore, progress in solving the stated problem will come from a study of Hardy numbers; in particular, from lower bounds for Hardy numbers. Chapter III is a step in this direction. Whereas the existing results are limited to functions whose image lies in a proper simply-connected subregion of $\mathscr{C}$, we give a lower bound for the Hardy number of an arbitrary region (Section 3). Some theorems of M. Tsuji play the central role here. The bound in Section 3 permits us to determine exactly the Hardy number of a starlike region (Section 4). We also derive a lower bound for the Hardy number of a simply-connected region whose boundary is sufficiently regular (Section 5). In some cases this is an improvement of the bound in Section 3, since it takes into account a rotational factor. Our tool here is Ahlfors' distortion theorem [1].

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