ON FIXED POINTS OF A COMPACT AUTOMORPHISM GROUP

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In this note, we investigate the existence of nontrivial fixed points under a compact, connected group of automorphisms of a Lie group. Although we cannot always hope for the existence of such fixed points in connected groups, the situation for nonconnected groups seems to be more favorable. The following theorem, the proof of which constitutes the main portion of this note, bears this out.

THEOREM 1. *Let* G *be a Lie group such that the identity component* G_0 *of* G *contains no nontrivial compact subgroup. Then, for every compact, connected group* C *of automorphisms of* G, *we have the relation* $G = G_0 \cdot F(C)$, *where* F(C) *denotes the collection of all fixed points of* C *in* G.

As an easy consequence of this theorem, we prove that a compact, connected group of automorphisms of G has a nontrivial fixed point, if G is a nonconnected Lie group that contains no compact semisimple subgroup. We also present other applications of the theorem, together with an example to supplement our result.

We note that the topology in the group of automorphisms of a Lie group is understood to be the so-called generalized compact-open topology, under which the group is a topological group. In the rest of this note, we use G_0 to denote the identity component of a Lie group G.

1. PROOF OF THEOREM 1

LEMMA 1. *Let* H *be a connected semisimple Lie group that contains no compact, semisimple subgroup. Then every compact, connected group of automorphisms of* H *is a torus.*

Proof. Let C be a compact, connected group of automorphisms of H, and let S be the commutator subgroup of C. We claim that S is trivial. Since H is semisimple, every automorphism in C is an inner automorphism. Since the adjoint group of H is isomorphic with H/Z, where Z denotes the center of H, we may identify S with a compact, connected subgroup of H/Z. With this identification, let L be the complete inverse image of S under the projection map $H \to H/Z$. Since Z is a discrete subgroup of H, L_0 is easily seen to be a covering group of S. Hence L_0 is a compact, connected semisimple subgroup of H. Because H contains no such subgroup by our hypothesis, it follows that L_0 is trivial. Hence S is trivial, and C is a torus.

LEMMA 2. *If* G_0 *is a semisimple group that contains no nontrivial compact subgroup, then the conclusion of Theorem* 1 *holds.*

Proof. Let π: $\mathrm{Aut}(G) \to \mathrm{Aut}(G_0)$ be the restricting homomorphism, where, for a Lie group L, Aut(L) denotes the group of automorphisms of L, and let $C_1 = \pi(C)$. Since π is continuous, C_1 is a compact subgroup of Aut(G). Choose a maximal

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