IMMERSIONS OF k-ORIENTABLE MANIFOLDS

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1. INTRODUCTION

Let M^m denote a smooth, closed, connected m-manifold. According to the classical theorems of Whitney, M^m embeds in R^{2m} and (if $m > 1$) immerses in R^{2m-1}. There are, however, many examples to show that the existence of an embedding $M^m \subset R^{2m-k+1}$ $(2 \leq k \leq m - 1)$ does not imply the existence of an immersion $M^m \subseteq R^{2m-k}$. In particular, complex projective space CP_m $(m = 2^r)$ embeds in R^{4m-1} [3] but does not immerse in R^{4m-2} [7]. In this note, we show that with additional restrictions, an embedding $M^m \subset R^{2m-k+1}$ will produce an immersion $M^m \subseteq R^{2m-k}$.

If α is a vector bundle over a CW-complex B, denote its stable equivalence class by (α). We say that (α) is *k-orientable* if the restriction of α to the k-skeleton of B is stably fibre-homotopy trivial. A manifold M^m (hereafter assumed to be smooth and connected) is k-orientable if its tangent bundle $\tau(M^m)$ is k-orientable. A map $f: M^m \to N^n$ between manifolds is *k-orientation-preserving* if $f^*(\tau(N^n)) - (\tau(M^m))$ is k-orientable. Let $i_0: N^n \to N^n \times R$ denote the inclusion $y \to (y, 0)$ $(y \in N^n)$. Our main result is the following.

THEOREM 1.1. *Suppose* $2k \leq m - 1$. *Let* M^m *be closed, and let*

$$f: M^m \to N^{2m-k}$$

be k-*orientation-preserving. If the composition* $i_0 f: M^m \to N^{2m-k} \times R$ *is homotopic to an embedding, then* f *is homotopic to an immersion.*

Some interesting corollaries follow.

COROLLARY 1.2. *Suppose* $2k \leq m - 1$. *Let* M^m *be closed and* k-*orientable. If* $M^m \subset R^{2m-k+1}$, *then* $M \subseteq R^{2m-k}$.

COROLLARY 1.3. *Suppose* $2k \leq m - 1$. *Let* $f: M^m \to N^{2m-k}$ *be given, where* M^m *is closed and* N^{2m-k} *is* k-*connected. Suppose either*

(a) M^m *is* k-*connected or*

(b) M^m *is* (k - 1)-*connected and* f *is* k-*orientation-preserving.*

Then f *is homotopic to an immersion.*

Proof. By A. Haefliger's embedding theorem [3], $i_0 f: M^m \to N^{2m-k} \times R$ is homotopic to an embedding. Now apply Theorem 1.1.

Note that, if M^m is (k - 1)-connected and $k \equiv 3, 5, 6,$ or $7 \pmod 8$, the assumption that f be k-orientation-preserving is superfluous. To verify this, let $\nu: M^m \to BO$ be a classifying map for $f^*(\tau(N^{2m-k})) - (\tau(M^m))$. There is a single obstruction to lifting ν to the k-connected covering BO[k] of BO. This occurs in

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