RATIONAL EXPRESSIONS OF CERTAIN AUTOMORPHIC FORMS

George W. Bright

1. Let G denote the group of fractional linear transformations of the upper half-plane $H^+ = \{x + iy \mid x, y \in \mathbb{R} \text{ and } y > 0\}$. For $\Gamma \subset G$, let f(z) be a Γ-automorphic form of weight k, and let (Γ, k) denote the vector space of all such forms. By considering generators of G, R. A. Rankin [1] and H. L. Resnikoff [3] have defined differential operators D^m (m is an integer exceeding 1) such that for all subgroups $\Gamma \subset G$, the relation

$$(1) \qquad D^m: (\Gamma, k) \to (\Gamma, m(k+2))$$

holds. Set $f_i(z) = \frac{d^i}{dz^i} f(z)$. It has been shown that if $f \in (\Gamma, h)$ and if

$$P \in \mathbb{C}[f, f_1, f_2, \cdots, f_m] \cap (\Gamma, k),$$

then P is a quotient of some Q in $\mathbb{C}[f, D^2 f, \cdots, D^m f]$ and an appropriate power of f(z).

We shall show that D^2 and D^3 are sufficient for a rational representation, if operator composition is admitted. Let "o" denote composition. If we define

$$(2) \qquad D^{r,s} f \equiv (D^3 \circ)^r \circ (D^2 \circ)^s f,$$

where r and s are integers, then it will be enough to show that $D^m f$ is a rational function of $\{D^{r,s} f\}$, for all pairs (r, s) such that

$$(3) \qquad r \in \{0, 1\}, \quad s \in \{0, 1, 2, \cdots, [m/2]\}, \quad \text{and } 3r + 2s \leq m.$$

Here, [x] denotes the greatest integer not exceeding x. The denominator of our expression will assume a convenient form.

2. Let $f(z) \in (\Gamma, k)$ $(k > 0)$, and denote d^m/dz^m by L^m. It is known [3] that

$$(4) \qquad L^m: (\Gamma, 1-m) \to (\Gamma, 1+m) \quad (m > 1).$$

An easy calculation shows that

$$f^{((k+1)m-1)/k} L^m(f^{(1-m)/k}) \in \mathbb{R}[f, f_1, \cdots, f_m].$$

That is, regardless of the branches chosen for $f^{((k+1)m-1)/k}$ and $f^{(1-m)/k}$, the final result is uniquely determined. Moreover, the resulting expression is in (Γ, m(k + 2)). Now define

Received July 21, 1969.

I should like to express my appreciation to Dr. H. L. Resnikoff for his help in suggesting and structuring the problem and to the referee for his assistance in improving the presentation.

Michigan Math. J. 17 (1970).