EXTREMAL LENGTH AS A CAPACITY

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1. INTRODUCTION

In Euclidean n-space E_n, the p-capacity $(1 \leq p < \infty)$ of a pair of disjoint closed sets C_0 and C_1 is defined as

$$\Gamma_p(C_0, C_1) = \inf\left(\int_{E_n} |\operatorname{grad} u|^p \, dL_n\right), \tag{1}$$

where the infimum is taken over all continuous functions u on E_n that are infinitely differentiable on $E_n - (C_0 \cup C_1)$ and assume values 0 on C_0 and 1 on C_1. Under the assumptions that C_0 contains the complement of some closed n-ball and that $1 < p < \infty$, it was shown in [14] that $\Gamma_p(C_0, C_1)$ is equal to the reciprocal of the p-dimensional extremal length of all continua in E_n that intersect both C_0 and C_1. This equality was first established by F. W. Gehring [10] in the case where p = n, and it plays an important role in the theory of quasiconformal mappings on E_n.

For an arbitrary set $E \subset E_n$, let $\psi_p(E)$ denote the reciprocal of the p-dimensional extremal length of all closed connected sets that join E to the point at infinity of E_n. By using the relationship between p-capacity and extremal length that was referred to above, we shall show that ψ_p is a capacity in the sense of Brelot.

Let W_p^1 denote the collection of distributions whose partial derivatives are functions locally in $\mathscr{L}^p$, and call a function u *p-precise* if $u \in W_p^1$ and if for every $\varepsilon > 0$, there exists an open set U such that $\psi_p(U) < \varepsilon$ and u restricted to the complement of U is continuous. For $p > 1$, we use the results of [8] to show that every function $u \in W_p^1$ is equivalent to a precise function, thus extending the result obtained by J. Deny and J. L. Lions [5] in the case p = 2. In the terminology of N. Aronszjan and K. Smith, the precise functions form a perfect functional completion whose exceptional sets are ψ_p-null sets. Finally, for every bounded Šuslin set $A \subset E_n$, we shall show that

$$\psi_p(A) = \inf\left(\int_{E_n} |\operatorname{grad} u|^p\right),$$

where the infimum is taken over all precise functions u that "vanish at infinity" and for which u(x) = 1 for ψ_p-almost all $x \in A$.

2. NOTATION AND PRELIMINARIES

By L_n and H^k, we denote n-dimensional Lebesgue measure and k-dimensional Hausdorff measure in E_n (for properties of the latter, see [6]). Let $\mathscr{L}^p$ be the

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