BACKWARD LOWER BOUNDS FOR SOLUTIONS OF MIXED PARABOLIC PROBLEMS

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1. A number of recent papers dealt with lower bounds for solutions of abstract first-order differential inequalities in Hilbert space, with applications to parabolic problems. P. J. Cohen and M. Lees [4] investigated an inequality of the form

$$\left|\frac{du}{dt} - Au\right| \leq \phi(t)\,|u|\,, \tag{1}$$

and assuming that A is symmetric and $\phi(t) \in L^p(0, \infty)$, they proved that $|u(t)| \geq K e^{-\mu t}$ for some constants K and μ. In [2], S. Agmon and L. Nirenberg improved these results. Agmon [1] then developed a unified approach to the convexity methods that made their debut in [2], and he made it possible to treat equations of the form (1), where A(t) satisfies rather weak conditions and, in particular, need not be self-adjoint.

One problem, which was pointed out by J.-L. Lions and B. Malgrange [6], is that Agmon's results do not apply to parabolic problems with lower-order terms whose order exceeds half the order of the elliptic operator corresponding to A(t). However, A. Friedman [5] established a forward uniqueness theorem for equations of the form

$$\left|\frac{1}{i}\frac{du}{dt} - A(t)u\right| \leq \eta\,|A(t)u| + K\,|u|\,, \tag{2}$$

where η is sufficiently small, which in applications placed no restriction on the lower-order terms. Friedman also obtained a "uniqueness at $-\infty$" result for such equations.

This paper is devoted to generalizations of Friedman's latter results. We obtain backward lower bounds in a higher norm for solutions of abstract equations in Hilbert space. In the applications to parabolic problems, the norm can be taken to be that in the Sobolev space $H^m(\Omega)$, where 2m is the order of the equation. Our assumptions about the equation and its solution are in most respects less restrictive than those imposed by Friedman. In particular, we need not suppose that the resolvent of A(t) exists.

2. Let H denote a complex Hilbert space with norm $|\cdot|$ and inner product $(\cdot, \cdot)$. We study H-valued functions u(t) that satisfy the vector differential equation

$$\frac{du}{dt} + A(t)u = f(t, u) \tag{3}$$

almost everywhere in (a, b). Here the H-valued function du/dt, defined almost everywhere on (a, b), denotes the derivative of u(t) in the distribution sense of [7], and, for almost all $t \in (a, b)$, we require that u(t) lie in the domain $\mathscr{D}(t)$ of the (in

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