LOCALLY PROJECTIVE SPACES OF DIMENSION ONE

Nicholaas H. Kuiper

Let $\mathbb{P}$ be the real projective line, and let $\tilde{\mathbb{P}}$ be the universal covering space of $\mathbb{P}$ with the lifted projective structure. Topologically, $\tilde{\mathbb{P}}$ is an interval. Every projective mapping of a neighborhood in $\tilde{\mathbb{P}}$ onto another neighborhood is the restriction of a unique projective homeomorphism of $\tilde{\mathbb{P}}$ onto $\tilde{\mathbb{P}}$. A projective transformation of $\mathbb{P}$ can be expressed with respect to homogeneous or nonhomogeneous preferred coordinates as follows:

homogeneous coordinates: $(X, Y) \mapsto (x^*, y^*) = (aX + bY, cX + dY)$, $ad - bc \neq 0$;

nonhomogeneous coordinates: $(X = xY) \mapsto x^* = \frac{ax + b}{cx + d}$.

A *locally projective space* $Z$ of dimension 1 is a manifold and a complete (that is, not properly contained in a larger) atlas of mutually compatible homeomorphisms, called *maps*, of neighborhoods in $Z$ onto neighborhoods in $\tilde{\mathbb{P}}$. Two maps $f: U \to U'$ and $g: V \to V'$ are *compatible* if, for each connected component $W$ of the intersection $U \cap V$, the mapping $gf^{-1}$ restricted to $f(W)$ is a projective transformation in $\tilde{\mathbb{P}}$.

Two maps in the atlas, both covering the point $z$ in $Z$, are said to be equivalent at $z$ if their restrictions to some neighborhood of $z$ coincide. Following Ehresmann [2], we call an equivalence class of maps in the atlas, all covering $z$, a *local jet*, and we denote it by $j_z$. A topology in the set of jets can be introduced by giving a base for the open sets as follows: the set of jets (that is, of equivalence classes of maps) which contain a map is an open set in the space of jets.

A connected component of the space of jets is a covering space of $Z$, with the projection $j: j_z \to z$. The mapping which sends a jet $j_z$ of this covering space into the image in $\tilde{\mathbb{P}}$ of the point $z$ under each of the maps of the equivalence class $j_z$ is a homeomorphism onto an interval in $\tilde{\mathbb{P}}$. This interval can therefore be considered as the universal covering space $\tilde{Z}$ with lifted locally projective structure of $Z$.

An analogous theorem and proof, given in [2], exist in the case of the locally homogeneous spaces, where the homogeneous space with a transitive Lie group of transformations takes the place of the projective line in the present article (see [1] to [9]).

There exist only two manifolds of dimension 1, in the topological sense: the open interval and the circle. We consider them separately.

*Case A.* $Z$ is topologically an interval. Here $\tilde{Z}$ is projectively equivalent with an interval in $\tilde{\mathbb{P}}$. The classification into projectively different cases is as follows:

A1. $Z = \tilde{\mathbb{P}}$.

A2. $Z$ is one of the two parts into which a point in $\tilde{\mathbb{P}}$ divides its complement in $\tilde{\mathbb{P}}$.

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