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ON THE INTUITIONISTIC EQUIVALENTIAL CALCULUS

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1 *Introduction* We consider first the fragment **ICE** of the intuitionistic propositional calculus which consists of all wffs in which the only connectives are C (implication) and E (equivalence). We then consider the fragment **IE** of this system. From the Gentzen system **GCE** corresponding to **ICE**, we construct a Gentzen system **GE** corresponding to **IE**, thus obtaining a characterization of **IE** which makes no reference to an implicational system. We then look at an axiomatization and, using **GE**, show that it does indeed constitute an axiom system for **IE**.

2 *The Systems* The system **ICE** is defined as follows: The wffs of **ICE** are those constructed of propositional variables and two binary connectives, C and E. The rules of inference are substitution and Modus Ponens (from P and CPQ we can derive Q). There are five axioms:

1) $CpCqp$
2) $CCpCqrCCpqCpr$
3) $CEpqCpq$
4) $CEpqCqp$
5) $CCpqCCqpEpq$.

We define **IE** to be the equivalential fragment of **ICE**. We now construct a Gentzen system **GCE** corresponding to **ICE**: A sequent of **GCE** is to be any expression of the form $P_1, \ldots, P_n \rightarrow Q$, where $P_1, \ldots, P_n$, and Q are wffs of **ICE**, and $n \geq 0$. An axiom of **GCE** is to be any sequent of the form $P \rightarrow P$. There are nine rules of inference, as follows (where Γ and Δ represent arbitrary sequences, possibly empty, of wffs of **ICE**):

$$C \rightarrow: \frac{\Gamma \rightarrow P \quad Q, \Gamma \rightarrow R}{CPQ, \Gamma \rightarrow R} \qquad \rightarrow C: \frac{P, \Gamma \rightarrow Q}{\Gamma \rightarrow CPQ}$$

$$E \rightarrow_1: \frac{\Gamma \rightarrow P \quad Q, \Gamma \rightarrow R}{EPQ, \Gamma \rightarrow R} \qquad E \rightarrow_2: \frac{\Gamma \rightarrow Q \quad P, \Gamma \rightarrow R}{EPQ, \Gamma \rightarrow R}$$

$$\rightarrow E: \frac{P, \Gamma \rightarrow Q \quad Q, \Gamma \rightarrow P}{\Gamma \rightarrow EPQ}$$

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