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SOME RESULTS CONCERNING FINITE MODELS FOR SENTENTIAL CALCULI

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Terminology and notation. Let $S_{\aleph_0}$ be the set of wffs built up in the usual way from denumerably many *letters* $p_1, p_2, \ldots$ and finitely many *connectives* $F_1, \ldots, F_n$ (each F_i a k_i-place connective for some positive integer k_i): letters are wffs, and $F_i\alpha_1 \ldots \alpha_{k_i}$ is a wff if $\alpha_1, \ldots, \alpha_{k_i}$ are wffs. A *rule of inference* is an s-tuple of wffs; and a set of wffs T is *closed under* a rule of inference $\langle\beta_1, \ldots, \beta_{s-1}, \beta_s\rangle$ just in case $\gamma_s \epsilon T$ whenever $\gamma_1, \ldots, \gamma_{s-1}, \gamma_s$ result from $\beta_1, \ldots, \beta_{s-1}, \beta_s$, respectively, by a uniform substitution of wffs for letters, and $\gamma_1, \ldots, \gamma_{s-1} \epsilon T$.

$\mathsf{P} = \langle T, A, R_1, \ldots, R_r\rangle$ is a *sentential calculus* if and only if A, the set of *axioms* of P, is a set of wffs, $R_1, \ldots, R_r$ are rules of inference, and T, the set of *theorems* of P, is the least set containing A and closed under substitution and each of $R_1, \ldots, R_r$. (Where $r = 0$, T is simply the set of substitution instances of members of A.) For each such P define an equivalence relation, $\cong_P$, on $S_{\aleph_0}$ by letting $\alpha \cong_P \beta$ just in case replacement of zero or more occurrences of α by β in each wff in T (respectively, not in T) results in a wff in T (respectively, not in T). For $\alpha \epsilon S \subset S_{\aleph_0}$, let $[\alpha] \cong_{P|S}$ be the set of β's in S such that $\alpha \cong_P \beta$ and let $S/\cong_P$ be the set of $[\alpha] \cong_{P|S}$'s such that $\alpha \epsilon S$.

$\mathfrak{M} = \langle V, D, f_1, \ldots, f_n\rangle$ is a *matrix* if and only if V is a non-empty set, $D \subset V$, and each f_i is a k_i-ary operation in V. A function h: $S_{\aleph_0} \to V$ is a *value function of* $\mathfrak{M}$ just in case $h(F_i\alpha_1 \ldots \alpha_{k_i}) = f_i(h(\alpha_1), \ldots, h(\alpha_{k_i}))$ for all $\alpha_1, \ldots, \alpha_{k_i} \epsilon S_{\aleph_0}$, and α is an *$\mathfrak{M}$-tautology* just in case $h(\alpha) \epsilon D$ for every value function h of $\mathfrak{M}$. We denote the set of $\mathfrak{M}$-tautologies by '$\mathrm{E}(\mathfrak{M})$'. Where $\mathfrak{M} = \langle V, D, f_1, \ldots, f_n\rangle$ and $\mathfrak{M}' = \langle V', D', f_1', \ldots, f_n'\rangle$ are matrices the matrix $\mathfrak{M} \times \mathfrak{M}' = \langle V \times V', D \times D', f_1^X, \ldots, f_n^X\rangle$, where $f_i^X(\langle v_1, v_1'\rangle, \ldots, \langle v_{k_i}, v_{k_i}'\rangle) = \langle f_i(v_1, \ldots, v_{k_i}), f_i'(v_1', \ldots, v_{k_i}')\rangle$, is called the *product* of $\mathfrak{M}$ and $\mathfrak{M}'$. Evidently (cf. [5]), $\mathrm{E}(\mathfrak{M} \times \mathfrak{M}') = \mathrm{E}(\mathfrak{M}) \cap \mathrm{E}(\mathfrak{M}')$.

The matrix $\mathfrak{M} = \langle V, D, f_1, \ldots, f_n\rangle$ is a *model* of the sentential calculus $\mathsf{P} = \langle T, A, R_1, \ldots, R_r\rangle$ if $T \subset \mathrm{E}(\mathfrak{M})$ and for each value function h of $\mathfrak{M}$ and each rule $\langle\beta_1, \ldots, \beta_{s-1}, \beta_s\rangle$ of P, if $h(\beta_1), \ldots, h(\beta_{s-1}) \epsilon D$ then $h(\beta_s) \epsilon D$. If $\mathfrak{M}$ is a model of P with $\mathrm{E}(\mathfrak{M}) = T$, we call $\mathfrak{M}$ a *characteristic* matrix for P.

For each set of letters L we let S_L be the set of wffs in which the only

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