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A NOTE ON SUZUKI'S CHAIN OF HYPERDEGREES

P. G. ODIFREDDI

In his very important work [5] Suzuki found some interesting results about Π_1^1 implicitly definable sets. Precisely he proved that (in the notations of Rogers [4] which we freely use)

1 *If* $\{A\} \in \Pi_1^1$ *and* $\{B\} \in \Pi_1^1$, *then*

1a $A \leq_h B$ *or* $B <_h A$

and

1b $A <_h B$ *iff* $\text{T}^A \leq_h B$ *iff* $\lambda^A < \lambda^B$

2 *If* $\{A\} \in \Pi_1^1$, *then* $\{\text{T}^A\} \in \Pi_1^1$.

Otherwise stated, the hyperdegrees of Π_1^1 implicitly definable sets are well-ordered in a chain $\{a_\alpha\}_{\alpha < \alpha_0}$ such that

3 a_0 *is the hyperdegree of* Δ_1^1 *sets*

and

4 $a_{\alpha+1} = a'_\alpha =$ *the hyperjump of* a_α.

Suzuki left open the characterization of α_0, that we now obtain* using some results of Moschovakis (see [4], p. 416):

Proposition α_0 *is* $\omega(\Delta_2^1)$, *that is the least ordinal which is not a* Δ_2^1*-ordinal.*

Proof: We split it in two parts:

(a) $\alpha_0 \leq \omega(\Delta_2^1)$. Given $\{A\} \in \Pi_1^1$ let w_A be a tree for A ([4], p. 432), that is $w_A \in \text{T}^X$ iff $X = A$. There exist a unique X (viz. A) s.t. $w_A \in \text{T}^X$, so that $\|w_A\|^2 = (\min_{w_A \in T^X} \|w_A\|^X) = \|w_A\|^A$. Then Lemma 1 of [4], p. 432, says that if $\{A\} \in \Pi_1^1$ and $\{B\} \in \Pi_1^1$ we have $\|w_A\|^2 \leq \|w_B\|^2 \Rightarrow A \leq_h B$, and by Suzuki's

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