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Algebraic Studies of First-Order Enlargements

Abraham Robinson in memoriam

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This paper may be considered as an axiomatic study of first-order elementary extensions and enlargements of *full* relational systems.

An axiomatization of this sort has been given by Robinson and Zakon [16] for superstructures which form a convenient set-theoretic framework for higher-order logic (cf. also Zakon [20], Keisler [8], Stroyan and Luxemburg [19], and Davis [3]). Most of Robinson's and Zakon's axioms (see (4.1)-(4.4) below) are only first order (Keisler [8]: elementary) in character. The language they use is that of a Boolean homomorphism α: $\mathcal{P}(A) \rightarrow \mathcal{P}(B)$; for our purposes we will frequently assume α to preserve just all finite intersections. Based on general lattice-theoretic considerations summarized in Section 0 (and Section 2), we will come up with five equivalent approaches in Sections 1-3. Two of these are the extension and contraction procedures (Section 1) well-known from general ring theory, less known from the theory of algebraic lattices. Another equivalent is the passing from a filter to its monad (Section 2), which constitutes one of the fundamental ideas in the exact foundation of Leibniz's infinitesimals discovered by Robinson [14]. A particularly striking equivalent approach seems that of an arbitrary mapping ω: $B \rightarrow \Phi(A)$ (lattice of filters on A). Booleanity of α: $\mathcal{P}(A) \rightarrow \mathcal{P}(B)$ being equivalent (see (2.12)) with $\omega[B] \subset \Omega(A)$ (set of ultrafilters on A), $\Omega(A) \subset \omega[B]$ characterizes (see (2.17) and (2.18)) enlargements in the sense of Robinson [14].

Sections 1-3 deal with the zero-order (Boolean) aspects of elementary extensions; for full first-order logic in Sections 4-6, we have to consider a sequence of at-least-finitely-intersection-preserving mappings α^n: $\mathcal{P}(A^n) \rightarrow$

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